

# Impulse Control of Multi-dimensional Jump Diffusions in Finite Time Horizon

Yann-Shin Aaron Chen <sup>\*</sup>      Xin Guo <sup>†</sup>

November 8, 2011

## Abstract

This paper analyzes a class of impulse control problems for multi-dimensional jump diffusions in a finite time horizon. Following the basic mathematical setup from Stroock and Varadhan [33], this paper first establishes rigorously an appropriate form of Dynamic Programming Principle (DPP). It then shows that the value function is a viscosity solution for the associated Hamilton-Jacobi-Bellman (HJB) equation involving integro-differential operators. Finally, it proves the  $W_{loc}^{(2,1),p}$  regularity for  $2 \leq p < \infty$  and the uniqueness of the viscosity solution.

**Keywords:** Stochastic Impulse Control, Viscosity solution, Parabolic Partial Differential Equations  
**AMS Classification:** 49J20, 49N25, 49N60

## 1 Introduction

This paper considers the following class of impulse control problem for an  $n$ -dimensional diffusion process  $X_t$ . In the absence of control,  $X_t$  is governed by an Itô's stochastic differential equation,

$$\begin{aligned} X_t = & x_0 + \int_{t_0}^t b(X_{s-}, s) ds + \int_{t_0}^t \sigma(X_{s-}, s) dW_s \\ & + \int_{t_0}^t \int j_1(X_{s-}, s, z) N(dz, dt) + \int_{t_0}^t \int j_2(X_{s-}, s, z) \tilde{N}(dz, dt), \end{aligned}$$

where  $W$  is a standard Brownian motion,  $\tilde{N} = N(dt, dz) - \rho(dz)dt$  with  $N$  being a Poisson point process on  $[0, T] \times \mathbb{R}^k$  with density  $\rho(dz)dt$ ,  $W$  and  $N$  are independent in an appropriate filtered probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , and  $b, \sigma, j_1, j_2$  satisfy suitable regularity conditions to be specified later. If a control policy  $V = (\tau_1, \xi_1; \tau_2, \xi_2; \dots)$  is adopted, then  $X_t$  evolves as

$$\begin{aligned} X_t = & x_0 + \int_{t_0}^t b(X_{s-}, s) ds + \int_{t_0}^t \sigma(X_{s-}, s) dW_s + \sum_i \xi_i 1_{(\tau_i \leq t)} \\ & + \int_{t_0}^t \int j_1(X_{s-}, s, z) N(dz, dt) + \int_{t_0}^t \int j_2(X_{s-}, s, z) \tilde{N}(dz, dt). \end{aligned}$$

Here the control  $(\tau_i, \xi_i)_i$  is of an impulse type such that  $\tau_1, \tau_2, \dots$  is an increasing sequence of stopping times with respect to  $\mathcal{F}_t^{W, N}$ , the natural filtration generated by  $W$  and  $N$ , and  $\xi_i$  is an  $\mathbb{R}^n$ -valued,  $\mathcal{F}_{\tau_i}^{W, N}$ -measurable random variable.

---

<sup>\*</sup>Department of Mathematics, University of California at Berkeley, CA 94720-3840. Email address: [yac@math.berkeley.edu](mailto:yac@math.berkeley.edu)

<sup>†</sup>Department of Industrial Engineering and Operations Research, University of California at Berkeley, CA 94720-1777. Email address: [xinguo@ieor.berkeley.edu](mailto:xinguo@ieor.berkeley.edu)

The objective is to choose an appropriate impulse control  $(\tau_i, \xi_i)_i$  so that the following cost function is minimized:

$$J[x_0, t_0, \tau_i, \xi_i] = \mathbb{E} \left[ \int_{t_0}^T f(X_t^{x_0, t_0, \tau_i, \xi_i}, t) dt + \sum_i B(\xi_i, \tau_i) 1_{\{t_0 \leq \tau_i \leq T\}} + g(X_T^{x_0, t_0, \tau_i, \xi_i}) \right].$$

Here  $f : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}$  is the running cost function,  $B : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}^n$  is the transaction cost function, and  $g : \mathbb{R}^n \times \mathbb{R}$  is the terminal cost function.

Impulse control, in contrast to regular and singular controls, allows the state space to be discontinuous and is a more natural mathematical framework for many applied problems in engineering and economics. Examples in financial mathematics include portfolio management with transaction costs [4, 22, 23, 11, 27, 29], insurance models [19, 7], liquidity risk [25], optimal control of exchange rates [20, 28, 8], and real options [35, 26]. Similar to their regular and singular control counterparts, impulse control problems can be analyzed via various approaches. One approach is to focus on solving for the value function the associated (quasi)-variational inequalities or Hamilton-Jacobian-Bellman (HJB) integro-differential equations, and then establishing the optimality of the solution by a verification type theorem. (See Øksendal and Sulem [30].) Another approach is to characterize the value function of the control problem as a (unique) viscosity solution to the associated PDEs, and/or to study their regularities. In both approaches, in order to connect the PDEs and the original control problems, a proper form of the Dynamic Programming Principle (DPP) is usually implicitly or explicitly assumed.

The earliest mathematical literature on impulse controls is the well-known book by Bensoussan and Lions [3], where value functions of the control problems for diffusions without jumps were shown to be the solutions of quasi-variational-inequalities and where their regularity properties were established when the control is strictly positive and the state space is in a bounded region. Recently, Barles, Chasseigne, and Imbert [1] provided a general framework and careful analysis on the unique viscosity solution of second order nonlinear elliptic/parabolic partial integro-differential equations. However, their paper did not study the DPP, i.e., the link between the PDEs and control problems. On the other hand, Tang and Yong [34] and Ishikawa [18] established some version of the DPP and the uniqueness of the viscosity solution for diffusions without jumps. More recently, Seydel [32] used a version of the DPP for Markov controls to study the viscosity solution of control problems on jump diffusions. Based on the restrictive setup of [32], Davis, Guo and Wu [10] focused on the regularities of the viscosity solution associated with the control problems on jump diffusions in an infinite time horizon. Their results used some of the techniques developed in Guo and Wu [17].

In essence, there are three aspects when studying impulse control problems: the DPP, the HJB, and its regularity. However, all previous work addressed only one or two of the above aspects and used quite different setups and assumptions, making it difficult to see exactly to what extent all the relevant properties hold in a given framework. This is the motivation of our paper.

**Our Results.** This paper studies the finite time horizon impulse control problem of Eqn. (2) on multi-dimensional jump diffusions of Eq. (1).

- First, we follow the classical setup of Stroock and Varadhan [33] and work on the natural filtration of the underlying Brownian motion and the Poisson process, instead of the “usual hypothesis”, i.e., the completed right continuous filtration adopted in previous work. Within this framework and based on the estimation techniques developed in Tang and Yong [34] for diffusion processes without jumps, we prove a general form of the DPP.

We remark that various forms of the DPP for impulse controls of jump diffusions have been exploited quite literally in the stochastic control literature, and their proofs can be found for several cases, yet not with the full generality needed in this paper. For instance, our result includes those in [34] and [32] as special cases and includes non-Markov controls. Because of the inclusion of the jumps in the diffusion processes and the possibility of non-Markov controls, there are essential mathematical subtlety and difficulties, hence the necessity to adopt the classical and framework of [33]. This framework ensures certain properties of the regular conditional probability, and ensures that the controlled jump diffusions are well defined. These properties are crucial for rigorously establishing the DPP. In a way, our approach to the DPP is in the similar spirit of Yong and Zhou [36] for one-dimensional regular controls.

Note that there are separate lines of research on the DPP, including the weak DPP formulation by Bouchard and Touzi [6] and Bouchard and Nutz [5], as well as the classical work by El Kaouri [12]. However, it does not seem easy for us to fit their results to our problem and setup.

- Second, we show that the value function is a viscosity solution in the sense of [1]. This form of viscosity solution is convenient for the HJB equations involving integro-differential operators, which is the key for analyzing control problems on jump diffusions.

Closely related to our work in this aspect are the works of [32] and [34]. The former allowed only Markov controls and the latter did not deal with jump diffusions.

- Third, we prove the  $W_{loc}^{(2,1),p}$  regularity and the unique viscosity solution properties for the value function with first-order jumps. Note that the uniqueness of the viscosity solution in our paper is a “local” uniqueness, which is appropriate to study the regularity property.

Compared to [10] for an infinite horizon problem, this paper is on a finite time horizon which requires different PDEs techniques. Moreover, [10] did not study the DPP, nor the uniqueness of the viscosity solution, and was restricted to Markov controls. Thus it built partial results in a restrictive setting. There were also studies by Xing and Bayraktar [2] and Pham [31] on value functions for optimal stopping problems for jump diffusions. Their work however did not involve controls.

To our best knowledge, our paper is the first that presents a comprehensive analysis for impulse control problems for jump diffusions: from the original control problem, to the related DPP, to the viscosity solution, and its uniqueness and regularity; all established under one mathematical setup.

## 2 Problem Formulation and Main Results

### 2.1 Problem formulation

**Filtration** Fix a time  $T > 0$ . For each  $t_0 \in [0, T]$ , let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space that supports a Brownian motion  $\{W\}_{t_0 \leq s \leq T}$  starting at  $t_0$ , and an independent Poisson point process  $N(dt, dz)$  on  $([t_0, T], \mathbb{R}^k \setminus \{0\})$  with intensity  $L \otimes \rho$ . Here  $L$  is the Lebesgue measure on  $[t_0, T]$  and  $\rho$  is a measure defined on  $\mathbb{R}^k \setminus \{0\}$ . For each  $t \in [t_0, T]$ , define  $\{\mathcal{F}_t^{W,N}\}_{t_0 \leq t \leq T}$  to be the natural filtration of the Brownian motion  $W$  and the Poisson process  $N$ , define  $\{\mathcal{F}_{t_0,t}[t_0, T]\}$  to be  $\{\mathcal{F}_t^{W,N}\}_{t_0 \leq t \leq T}$  restricted to the interval  $[t_0, t]$ .

Throughout the paper, we will use this uncompleted natural filtration  $\{\mathcal{F}_{t_0,t}[t_0, T]\}$ . This specification ensures that the stochastic integration and therefore the controlled jump diffusion to be well defined. (See Lemma 4.3.3 from Stroock & Varadhan [33]).

Now, we can define mathematically the impulse control problem, starting with the set of admissible controls.

**Definition 1.** *The set of admissible impulse control  $\mathcal{V}[t_0, T]$  consists of pairs of sequences  $\{\tau_i, \xi_i\}_{1 \leq i < \infty}$  such that*

1.  $\tau_i : \Omega \rightarrow [t_0, T] \cup \{\infty\}$  such that  $\tau_i$  are stopping times with respect to the filtration  $\{\mathcal{F}_{t_0,s}^{W,N}\}_{t_0 \leq s \leq T}$ ,
2.  $\tau_i \leq \tau_{i+1}$  for all  $i$ ,
3.  $\xi_i : \Omega \rightarrow \mathbb{R}^n \setminus \{0\}$  is a random variable such that  $\xi_i \in \mathcal{F}_{t_0,\tau_i}^{W,N}$ .

Now, given an admissible impulse control  $\{\tau_i, \xi_i\}_{1 \leq i < \infty}$ , a stochastic process  $(X_t)_{t \geq 0}$  follows a stochastic differential equation with jumps,

$$\begin{aligned} X_t = & x_0 + \int_{t_0}^t b(X_{s-}, s) ds + \int_{t_0}^t \sigma(X_{s-}, s) dW_s + \sum_i \xi_i 1_{(\tau_i \leq t)} \\ & + \int_{t_0}^t \int j_1(X_{s-}, s, z) N(dz, dt) + \int_{t_0}^t \int j_2(X_{s-}, s, z) \tilde{N}(dz, dt). \end{aligned} \quad (1)$$

Here  $\tilde{N} = N(dt, dz) - \rho(dz)dt$ ,  $b : \mathbb{R}^n \times [0, T] \times \mathbb{R}^n$ ,  $\sigma : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}^{n \times m}$ , and  $j_1, j_2 : \mathbb{R}^n \times [0, T] \times \mathbb{R}^k \rightarrow \mathbb{R}^n$ . For each  $(\tau_i, \xi_i)_i \in \mathcal{V}[t_0, T]$ , and  $(x_0, t_0) \in \mathbb{R}^n \times [t_0, T]$ , denote  $X = X^{t_0, x_0, u, \tau_i, \xi_i}$ .

The stochastic control problem is to

$$\text{(Problem)} \quad \text{Minimize} \quad J[x_0, t_0, \tau_i, \xi_i] \quad \text{over all } (\tau_i, \xi_i) \in \mathcal{V}[t, T], \quad (2)$$

subject to Eqn. (1) with

$$J[x_0, t_0, \tau_i, \xi_i] = \mathbb{E} \left[ \int_{t_0}^T f(s, X_s^{t_0, x_0, \tau_i, \xi_i}) ds + g(X_T^{t_0, x_0, \tau_i, \xi_i}) \right] + \mathbb{E} \left[ \sum_i B(\xi_i, \tau_i) 1_{\{t_0 \leq \tau_i \leq T\}} \right]. \quad (3)$$

Here we denote  $V$  for the associated value function

$$\text{(Value Function)} \quad V(x, t) = \inf_{(\tau_i, \xi_i) \in \mathcal{V}[t, T]} J[x, t, \tau_i, \xi_i]. \quad (4)$$

In order for  $J$  and  $V$  to be well defined, and for the Brownian motion  $W$  and the Poisson process  $N$  as well as the controlled jump process  $X^{x_0, t_0, \tau_i, \xi_i}$  to be unique at least in a distribution sense, we shall specify some assumptions in Section 2.2.

The focus of the paper is to analyze the following HJB equation associated with the value function

$$\text{(HJB)} \quad \begin{cases} \max\{-u_t + Lu - f - Iu, u - Mu\} = 0 & \text{in } \mathbb{R}^n \times (0, T), \\ u = g & \text{on } \mathbb{R}^n \times \{t = T\}. \end{cases}$$

Here

$$\begin{aligned} I\phi(x, t) &= \int \phi(x + j_1(x, t, z), t) - \phi(x, t) \rho(dz) \\ &+ \int \phi(x + j_2(x, t, z), t) - \phi(x, t) - D\phi(x, t) \cdot j_2(x, t, z) \rho(dz), \end{aligned} \quad (5)$$

$$Mu(x, t) = \inf_{\xi \in \mathbb{R}^n} (u(x + \xi, t) + B(\xi, t)), \quad (6)$$

$$Lu(x, t) = -\text{tr} [A(x, t) \cdot D^2 u(x, t)] - b(x, t) \cdot Du(x, t) + ru(x, t). \quad (7)$$

**Main result.** Our main result states that the value function  $V(x, t)$  is a unique  $W_{loc}^{(2,1),p}(\mathbb{R}^n \times (0, T))$  viscosity solution to the (HJB) equation with  $2 \leq p < \infty$ . In particular, for each  $t \in [0, T]$ ,  $V(\cdot, t) \in C_{loc}^{1,\gamma}(\mathbb{R}^n)$  for any  $0 < \gamma < 1$ .

The main result is established in three steps.

- First, in order to connect the (HJB) equation with the value function, we prove an appropriate form of the DPP. (Theorem 1).
- Then, we show that the value function is a continuous viscosity solution to the (HJB) equation in the sense of [1]. (Theorem 2).
- Finally, we show that the value function is  $W_{loc}^{(2,1),p}$  for  $2 \leq p < \infty$ , and in fact a unique viscosity solution to the (HJB) equation. (Theorem 6).

All results, unless otherwise specified, are built under the assumptions specified in Section 2.2.

## 2.2 Outstanding assumptions

**Assumption 1.** Given  $t_0 \leq T$ , assume that

$$\begin{aligned} (\Omega_{t_0, T}, \mathcal{F}, \{\mathcal{F}_{t_0, t}[t_0, T]\}_{t_0 \leq t \leq T}) &= (C[t_0, T] \times M[t_0, T], \\ &\mathcal{B}_{t_0, T}[t_0, T] \otimes \mathcal{M}_{t_0, T}[t_0, T], \\ &\{\mathcal{B}_{t_0, t}[t_0, T] \otimes \mathcal{M}_{t_0, t}[t_0, T]\}_{t_0 \leq t \leq T}) \end{aligned}$$

such that the projection map  $(W, N)(x, n) = (x, n)$  is the Brownian motion and the Poisson point process with density  $\rho(dz) \times dt$  under  $\mathbb{P}$ , and for  $t_0 \leq s \leq t \leq T$ ,

$$\begin{aligned} C[t_0, T] &= \{x : [t_0, T] \rightarrow \mathbb{R}^n, x_{t_0} = 0\}, \\ M[t_0, T] &= \text{the class of locally finite measure on } [t_0, T] \times \mathbb{R}^k \setminus \{0\}, \\ \mathcal{B}_{s,t}[t_0, T] &= \sigma(\{x_r : x_r \in C[t_0, T], s \leq r \leq t\}), \\ \mathcal{M}_{s,t}[t_0, T] &= \sigma(\{n(B) : B \in \mathcal{B}([s, t] \times \mathbb{R}^k \setminus \{0\}), n \in M[t_0, T]\}). \end{aligned}$$

**Assumption 2.** (*Lipschitz Continuity.*) The functions  $b$ ,  $\sigma$ , and  $j$  are deterministic measurable functions such that there exists constant  $C$  independent of  $t \in [t_0, T]$ ,  $z \in \mathbb{R}^k \setminus \{0\}$  such that

$$\begin{aligned} |b(x, t) - b(y, t)| &\leq C|x - y|, \\ |\sigma(x, t) - \sigma(y, t)| &\leq C|x - y|, \\ \int_{|z| \geq 1} |j_1(x, t, z) - j_1(y, t, z)| \rho(dz) &\leq C|x - y|, \\ \int_{|z| < 1} |j_2(x, t, z) - j_2(y, t, z)|^2 \rho(dz) &\leq C|x - y|^2. \end{aligned}$$

**Assumption 3.** (*Growth Condition.*) There exists constant  $C > 0$ ,  $\nu \in [0, 1]$ , such that for any  $x, y \in \mathbb{R}^n$ ,

$$\begin{aligned} |b(t, x)| &\leq L(1 + |x|^\nu), \\ |\sigma(t, x)| &\leq L(1 + |x|^{\nu/2}), \\ \int_{|z| \geq 1} |j_1(x, t, z)| \nu(dz) &\leq C(1 + |x|^\nu), \\ \int_{|z| < 1} |j_2(x, t, z)|^2 \nu(dz) &\leq C(1 + |x|^\nu). \end{aligned}$$

**Assumption 4.** (*Hölder Continuity.*)  $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}$  and  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  are measurable functions such that there exists  $C > 0$ ,  $\delta \in (0, 1]$ ,  $\gamma \in (0, \infty)$  such that

$$\begin{aligned} |f(t, x) - f(t, \hat{x})| &\leq C(1 + |x|^\gamma + |\hat{x}|^\gamma)|x - \hat{x}|^\delta, \\ |g(x) - g(\hat{x})| &\leq C(1 + |x|^\gamma + |\hat{x}|^\gamma)|x - \hat{x}|^\delta, \end{aligned}$$

for all  $t \in [0, T]$ ,  $x, \hat{x} \in \mathbb{R}^n$ .

**Assumption 5.** (*Lower Boundedness*) There exists an  $L > 0$  and  $\mu \in (0, 1]$  such that

$$\begin{aligned} f(t, x) &\geq -L, \\ h(x) &\geq -L, \\ B(\xi, t) &\geq L + C|\xi|^\mu, \end{aligned}$$

for all  $t \in [0, T]$ ,  $x \in \mathbb{R}^n$ ,  $\xi \in \mathbb{R}^n$ .

**Assumption 6.** (*Monotonicity and Subadditivity*)  $B : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}$  is a continuous function such that for any  $0 \leq s \leq t \leq T$ ,  $B(t, \xi) \leq B(s, \xi)$ , and for  $(t, \xi), (t, \hat{\xi})$  being in a fixed compact subset of  $\mathbb{R}^n \times [0, T]$ , there exists constant  $K > 0$  such that

$$B(t, \xi + \hat{\xi}) + K \leq B(t, \xi) + B(t, \hat{\xi}).$$

**Assumption 7.** (*Dominance*) The growth of  $B$  exceeds the growth of the cost functions  $f$  and  $g$  so that

$$\begin{aligned} \delta + \gamma &< \mu, \\ \nu &\leq \mu. \end{aligned}$$

**Assumption 8.** (No Terminal Impulse) For any  $x, \xi \in \mathbb{R}^n$ ,

$$g(x) \leq \inf_{\xi} g(x + \xi) + B(\xi, T).$$

**Assumption 9.** Suppose that there exists a measurable map  $M : \mathbb{R}^n \times [0, T] \rightarrow M(\mathbb{R}^n \setminus \{0\})$ , in which  $M(\mathbb{R}^n \setminus \{0\})$  is the set of locally finite measure on  $\mathbb{R}^n \setminus \{0\}$ , such that one has the following representation of the integro operator:

$$I\phi(x, t) = \int [\phi(x + z, t) - \phi(x, t) - D\phi(x, t) \cdot z 1_{|z| \leq 1}] M(x, t, dz).$$

And assume that for  $(x, t)$  in some compact subset of  $\mathbb{R}^n \times [0, T]$ , there exists  $C$  such that

$$\int_{|z| < 1} |z|^2 M(x, t, dz) + \int_{|z| \geq 1} |z|^{\gamma + \delta} M(x, t, dz) \leq C.$$

**Notations** Throughout the paper, unless otherwise specified, we will use the following notations.

- $0 < \alpha \leq 1$ .

- 

$$A(x, t) = (a_{ij})_{n \times n}(x, t) = \frac{1}{2} \sigma(x, t) \sigma(x, t)^T.$$

- $\Xi(x, t)$  is the set of points  $\xi$  for which  $MV$  achieves the value, i.e.,

$$\Xi(x, t) = \{\xi \in \mathbb{R}^n : MV(x, t) = V(x + \xi, t) + B(\xi, t)\}.$$

- The continuation region  $\mathcal{C}$  and the action region  $\mathcal{A}$  are

$$\mathcal{C} := \{(x, t) \in \mathbb{R}^n \times [0, T] : V(x, t) < MV(x, t)\}, \quad (8)$$

$$\mathcal{A} := \{(x, t) \in \mathbb{R}^n \times [0, T] : V(x, t) = MV(x, t)\}. \quad (9)$$

- Let  $\Omega$  be a bounded open set in  $\mathbb{R}^{n+1}$ . Denote  $\partial_P \Omega$  to be the parabolic boundary of  $\Omega$ , which is the set of points  $(x_0, t_0) \in \overline{\Omega}$  such that for all  $R > 0$ ,  $Q(x_0, t_0; R) \not\subset \overline{\Omega}$ . Here  $Q(x_0, t_0; R) = \{(x, t) \in \mathbb{R}^{n+1} : \max\{|x - x_0|, |t - t_0|^{1/2}\} < R, t < t_0\}$ .

Note that  $\overline{\Omega}$  is the closure of the open set  $\Omega$  in  $\mathbb{R}^{n+1}$ . In the special case of a cylinder,  $\Omega = Q(x_0, t_0; R)$ , the parabolic boundary  $\partial_P \Omega = (B(0, R) \times \{t = 0\}) \cup (\{|x| = R\} \times [0, T])$ .

- Function spaces for  $\Omega$  being a bounded open set,

$$W^{(1,0),p}(\Omega) = \{u \in L^p(\Omega) : u_{x_i} \in L^p(\Omega)\},$$

$$W^{(2,1),p}(\Omega) = \{u \in W^{(1,0),p}(\Omega) : u_{x_i x_j} \in L^p(\Omega)\},$$

$$C^{2,1}(\overline{\Omega}) = \{u \in C(\overline{\Omega}) : u_t, u_{x_i x_j} \in C(\overline{\Omega})\},$$

$$C^{0+\alpha, 0+\frac{\alpha}{2}}(\overline{\Omega}) = \{u \in C(\overline{\Omega}) : \sup_{(x,t),(y,s) \in \Omega, (x,t) \neq (y,s)} \frac{|u(x,t) - u(y,s)|}{(|x-y|^2 + |t-s|)^{\alpha/2}} < \infty\},$$

$$C^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{\Omega}) = \{u \in C(\overline{\Omega}) : u_{x_i x_j}, u_t \in C^{0+\alpha, 0+\frac{\alpha}{2}}(\overline{\Omega})\},$$

$$L_{loc}^p(\Omega) = \{u|_U \in L^p(U) \ \forall \text{ open } U \text{ such that } \overline{U} \subset \overline{\Omega} \setminus \partial_P \Omega\},$$

$$W_{loc}^{(1,0),p}(\Omega) = \{u \in L_{loc}^p(\Omega) : u \in W^{(1,0),p}(U) \ \forall \text{ open } U \text{ such that } \overline{U} \subset \overline{\Omega} \setminus \partial_P \Omega\},$$

$$W_{loc}^{(2,1),p}(\Omega) = \{u \in L_{loc}^p(\Omega) : u \in W^{(2,1),p}(U) \ \forall \text{ open } U \text{ such that } \overline{U} \subset \overline{\Omega} \setminus \partial_P \Omega\}.$$

### 3 Dynamic Programming Principle and Some Preliminary Results

#### 3.1 Dynamic Programming Principle

**Theorem 1. (Dynamic Programming Principle)** *Assuming (A1), (A2), (A3), (A4), and (A5). For  $t_0 \in [0, T]$ ,  $x_0 \in \mathbb{R}^n$ , let  $\tau$  be a stopping time on  $(\Omega_{t_0, T}, \{\mathcal{F}_{t_0, t}\}_t)$ , we have*

$$V(t_0, x_0) = \inf_{(\tau_i, \xi_i) \in \mathcal{V}[t_0, T]} \mathbb{E} \left[ \int_{t_0}^{\tau \wedge T} f(s, X_s^{t_0, x_0, \tau_i, \xi_i}) ds \right] + \mathbb{E} \left[ \sum_i B(\xi_i, \tau_i) 1_{\tau_i \leq \tau \wedge T} + V(\tau \wedge T, X_{\tau \wedge T}^{t_0, x_0, \tau_i, \xi_i}) \right]. \quad (10)$$

In order to establish the DPP, the first key issue is: given a stopping time  $\tau$ , how the martingale property and the stochastic integral change under the regular conditional probability distribution  $(\mathbb{P}|\mathcal{F}_\tau)$ . The next key issue is the continuity of the value function, which will ensure that a countable selection is adequate without the abstract measurable selection theorem. (See [14]).

To start, let us first introduce a new function that connects two Brownian paths which start from the origin at different times into a single Brownian path. This function also combines two Poisson measures on different intervals into a single Poisson measure.

**Definition 2.** *For each  $t \in [t_0, T]$ , define a map  $\Pi^t = (\Pi_1^t, \Pi_2^t) : C[t_0, T] \times M[t_0, T] \rightarrow C[t_0, t] \times M[t_0, t] \times C[t, T] \times M[t, T]$  such that*

$$\begin{aligned} \Pi_1^t(x, n) &= (x|_{[t_0, t]}, n|_{[t_0, t] \times \mathbb{R}^k \setminus \{0\}}), \\ \Pi_2^t(x, n) &= (x|_{[t, T]} - x_t, n|_{(t, T] \times \mathbb{R}^k \setminus \{0\}}). \end{aligned}$$

Note that this is an  $\mathcal{F}_{t_0, T}[t_0, T]/\mathcal{F}_{t_0, t}[t_0, t] \otimes \mathcal{F}_{t, T}[t, T]$ -measurable bijection. Therefore, for fixed  $(y, m) \in C[t_0, T] \times M[t_0, T]$ , the map from  $C[t, T] \times M[t, T] \rightarrow C[t_0, T] \times M[t_0, T]$  defined by

$$\begin{aligned} (x, n) &\mapsto (\Pi^t)^{-1}(\Pi_1^t(y, m), \Pi_2^t(x, n)) \\ &= (x_{\cdot \vee t} - x_t + y_{\cdot \wedge t}, m|_{[t_0, t] \times \mathbb{R}^k \setminus \{0\}} + n|_{(t, T] \times \mathbb{R}^k \setminus \{0\}}) \end{aligned}$$

is  $\mathcal{F}_{t, s}[t, T]/\mathcal{F}_{t_0, s}[t_0, T]$ -measurable for each  $s \in [t, T]$ .

Next, we need two technical lemmas regarding  $(\mathbb{P}|\mathcal{F}_\tau)$ . Specifically, the first lemma states that the local martingale property is preserved, and the second one ensures that the stochastic integration is well defined under  $(\mathbb{P}|\mathcal{F}_\tau)$ .

According to Theorem 1.2.10 of [33],

**Lemma 1.** *Given a filtered space,  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$ , and an associated martingale  $\{M_t\}_{0 \leq t \leq T}$ . Let  $\tau$  be an  $\mathcal{F}$ -stopping time. Assume  $(\mathbb{P}|\mathcal{F}_\tau)$  exists. Then, for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ ,  $N_t = M_t - M_{t \wedge \tau}$  is a local martingale under  $(\mathbb{P}|\mathcal{F}_\tau)(\omega, \cdot)$ .*

**Lemma 2.** *Given a filtered space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$ , a stopping time  $\tau$ , a previsible process  $H : (0, T] \times \Omega \rightarrow \mathbb{R}^n$ , a local martingale  $M : [0, T] \times \Omega \rightarrow \mathbb{R}^n$  such that*

$$\int_\tau^T |H_s|^2 d[M]_s < \infty$$

$\mathbb{P}$ -almost surely, and  $N_t = \int_\tau^t H_s dM_s$  (a version of the stochastic integral that is right-continuous on all paths). Assume that  $(\mathbb{P}|\mathcal{F}_\tau)$  exists. Then, for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ ,  $N_t$  is also the stochastic integral  $\int_\tau^t H_s dM_s$  under the new probability measure  $(\mathbb{P}|\mathcal{G})(\omega, \cdot)$ .

The proof is elementary and is listed in the Appendix for completeness.

Now, we establish the first step of the Dynamic Programming Principle.

**Proposition 1.** Let  $\tau$  be a stopping time defined on some setup  $(\Omega, \{\mathcal{F}_{t_0,s}\})$ . For any impulse control  $(\tau_i, \xi_i) \in \mathcal{V}[t_0, T]$ ,

$$\begin{aligned} J[t_0, x_0, \tau_i, \xi_i] = & \mathbb{E} \left[ \int_{t_0}^{\tau \wedge T} f(s, X_s^{t_0, x_0, \tau_i, \xi_i}) ds + \sum_i B(\xi_i, \tau_i) 1_{\tau_i < \tau \wedge T} \right] \\ & + \mathbb{E} \left[ J[\tau \wedge T, X_{\tau \wedge T-}^{t_0, x_0, \tau_i^\omega, \xi_i^\omega}, \tau_i^\omega, \xi_i^\omega] \right]. \end{aligned} \quad (11)$$

Here  $\tau_i^\omega, \xi_i^\omega$  are defined as follows. For  $t \in [t_0, T]$ , for each  $(y, m) \in C[t_0, T] \times M([t_0, T] \times \mathbb{R}^k \setminus \{0\})$ ,

$$\begin{aligned} \tau_i^{y, \cdot, m}(x, n) &= \tau_i((\Pi^t)^{-1}(\Pi_1^t(y, n), \Pi_2^t(x, n))), \\ \xi_i^{y, \cdot, m}(x, n) &= \xi_i((\Pi^t)^{-1}(\Pi_1^t(y, n), \Pi_2^t(x, n))). \end{aligned}$$

And for each  $\omega$ ,

$$\begin{aligned} \tau_i^\omega &= \tau_i^{\tau(\omega), W(\omega), N(\omega)}, \\ \xi_i^\omega &= \xi_i^{\tau(\omega), W(\omega), N(\omega)}. \end{aligned}$$

*Proof.* Consider  $(\mathbb{P}|\mathcal{F}_{t_0, \tau})$  on  $(\Omega_{t_0, t}, \{\mathcal{F}_{t_0, t}\})$ . Since we are working with canonical spaces, the sample space is in fact a Polish space (see [21] Theorem A2.1 and A2.3), and the regular conditional probability exists by Theorem 6.3 of [21]. Since Polish spaces are completely separable metric spaces and have countably generated  $\sigma$ -algebra,  $\mathcal{F}_{t_0, \tau}$  is countably generated. By Lemma 1.3.3 from Stroock & Varadhan [33], there exists some null set  $N_0$  such that if  $(x, n) \notin N_0$ , then

$$(\mathbb{P}|\mathcal{F}_{t_0, \tau})((x, n), \{(y, n) : \Pi_1^{\tau(x, n)}(y, n) = \Pi_1^{\tau(x, n)}(x, n)\}) = 1.$$

Therefore, for  $\omega = (x, n) \notin N_0$ ,  $\tau_i = \tau_i^\omega$ , and  $\xi_i = \xi_i^\omega$  almost surely.

Moreover, by Lemma 2, the stochastic integrals are preserved. Therefore, for  $\omega \notin N_0$ , the solution to Eq. (1) remains a solution to the same equation on the interval  $[\tau(\omega), T]$  with  $(\tau_i^\omega, \xi_i^\omega) \in \mathcal{V}[\tau(\omega), T]$ . So  $X^{t_0, x_0, \tau_i, \xi_i}$  on the interval  $[\tau(\omega), T]$  has the same distribution as  $X^{\tau(\omega), y, \tau_i^\omega, \xi_i^\omega}$  for  $y = X_{\tau(\omega)}^{t_0, x_0, u, \cdot, \tau_i, \xi_i}(\omega)$  under  $(\mathbb{P}|\mathcal{F}_{t_0, \tau})(\omega, \cdot)$  for  $\omega \notin N_0$ .  $\square$

Now, to obtain the Dynamic Programming Principle, one needs to take the infimum on both sides of Eq. (11). The part of “ $\leq$ ” is immediate, but the opposite direction is more delicate. At the stopping time  $\tau$ , for each  $\omega$ , one needs to choose a good control so that the cost  $J$  is close to the optimal  $V$ . To do this, one needs to show that the functional  $J$  is continuous in some sense, and therefore a countable selection is adequate.

The following result, the Hölder continuity of the value function, is essentially Theorem 3.1 of Tang & Yong [34]. The major difference is that their work is for diffusions without jumps, therefore some modification in terms of estimation and adaptedness are needed, as outlined in the proof.

**Lemma 3.** There exists constant  $C > 0$  such that for all  $t, \hat{t} \in [0, T]$ ,  $x, \hat{x} \in \mathbb{R}^n$ ,

$$\begin{aligned} -C(T+1) &\leq V(t, x) \leq C(1 + |x|^{\gamma+\delta}), \\ |V(t, x) - V(\hat{t}, \hat{x})| &\leq C[(1 + |x|^\mu + |\hat{x}|^\mu)|t - \hat{t}|^{\delta/2} + (1 + |x|^\gamma + |\hat{x}|^\gamma)|x - \hat{x}|^\delta]. \end{aligned}$$

*Proof.* To include the jump terms, it suffices to note the following inequalities,

$$\begin{aligned} \mathbb{E} \left| \int_{t_0}^t \int j_1(s, X_s, z) N(dz, ds) \right|^\beta &\leq \mathbb{E} \left( \int_{t_0}^t \int |j_1(s, X_s, z)| \rho(dz) ds \right)^\beta, \\ \mathbb{E} \left| \int_{t_0}^t \int j_2(s, X_s, z) \tilde{N}(dz, ds) \right|^\beta &\leq \mathbb{E} \left( \int_{t_0}^t \int |j_2(s, X_s, z)|^2 \rho(dz) ds \right)^{\beta/2}. \end{aligned}$$

Moreover, in our framework,  $\bar{\xi}(\cdot)$  and  $\hat{\xi}(\cdot)$  would not be in  $\mathcal{V}[\hat{t}, T]$  because it is adapted to the filtration  $\{\mathcal{F}_{t,s}^{W,N}\}_{\hat{t} \leq s \leq T}$  instead of  $\{\mathcal{F}_{t,s}^{W,N}\}_{\hat{t} \leq s \leq T}$ . To fix this, consider for each  $\omega \in \Omega_{t_0, T}$ ,

$$\begin{aligned} \bar{\xi}^\omega(\cdot) &= \bar{\xi}((\Pi^{\hat{t}})^{-1}(\Pi_1^{\hat{t}}(\omega), \Pi_2^{\hat{t}}(\cdot))), \\ \hat{\xi}^\omega(\cdot) &= \hat{\xi}((\Pi^{\hat{t}})^{-1}(\Pi_1^{\hat{t}}(\omega), \Pi_2^{\hat{t}}(\cdot))), \end{aligned}$$



and consequently use  $\mathbb{E}[J[\hat{t}, x, \bar{\xi}^\omega]]$  instead of  $J[\hat{t}, x, u, \bar{\xi}]$ .  $\square$

Given that the value function  $V$  is continuous, we can prove Theorem 1.

*Proof.* (Dynamic Programming Principle) Without loss of generality, assume that  $\tau \leq T$ .

$$\begin{aligned} J[t_0, x_0, \tau_i, \xi_i] &= \mathbb{E} \left[ \int_{t_0}^{\tau} f(s, X_s^{t_0, x_0}) ds + \sum_i B(\tau_i, \xi_i) 1_{\tau_i < \tau} \right] \\ &\quad + \mathbb{E} \left[ J[\tau, X_{\tau-}^{t_0, x_0, u_{\tau-}^\omega, \tau_i^\omega, \xi_i^\omega}, u_{\tau-}^\omega, \tau_i^\omega, \xi_i^\omega] \right] \\ &\geq \mathbb{E} \left[ \int_{t_0}^{\tau} f(s, X_s^{t_0, x_0, \tau_i^\omega, \xi_i^\omega}) ds + \sum_i B(\tau_i, \xi_i) 1_{\tau_i < \tau} \right] \\ &\quad + \mathbb{E} \left[ V(\tau-, X_{\tau-}^{t_0, x_0, \tau_i^\omega, \xi_i^\omega}) \right]. \end{aligned}$$

Taking infimum on both sides, we get

$$V(t_0, x_0) \geq \inf_{(\tau_i, \xi_i) \in \mathcal{V}_{t_0}} \mathbb{E} \left[ \int_{t_0}^{\tau} f(s, X_s^{t_0, x_0, \tau_i^\omega, \xi_i^\omega}) ds + \sum_i B(\tau_i, \xi_i) 1_{\tau_i \leq \tau} + V(\tau, X_{\tau}^{t_0, x_0, \tau_i^\omega, \xi_i^\omega}) \right].$$

Now we are to prove the reverse direction for the above inequality.

Fix  $\epsilon > 0$ . Divide  $\mathbb{R}^n \times [t_0, T)$  into rectangles  $\{R_j \times [s_j, t_j)\}$  disjoint up to boundaries, such that for any  $x, \hat{x} \in R_j$  and  $t, \hat{t} \in [s_j, t_j)$ ,

$$\begin{aligned} |V(x, t) - V(\hat{x}, \hat{t})| &< \epsilon, \\ |t_j - s_j| &< \epsilon, \\ \text{diam}(R_j) &< \epsilon. \end{aligned}$$

For each  $R_j$ , pick  $x_j \in R_j$ . For each  $(x_j, t_j)$ , choose  $u^j \in \mathcal{U}[t_j, T]$ ,  $(\tau_k^j, \xi_k^j) \in \mathcal{V}[t_j, T]$ , such that  $V(x_j, t_j) + \epsilon > J[x_j, t_j, \tau_i^j, \xi_i^j]$ . Let

$$A_j = \{(X_{\tau_j}^{t_0, x_0, \tau_i, \xi_i}, \tau^j) \in R_j \times (s_j, t_j)\}$$

Note that  $\{A_j\}_j$  partitions the sample space  $C[t_0, T] \times M[t_0, T]$ . Define a new stopping time  $\hat{\tau}$  by:

$$\hat{\tau} = t_j \quad \text{on } A_j.$$

Note that  $\hat{\tau} > \tau$  unless  $\hat{\tau} = T$ .

Define a new strategy  $(\hat{\tau}_i, \hat{\xi}_i) \in \mathcal{V}[t_0, T]$  by the following,

$$\sum_i \hat{\xi}_i 1_{\hat{\tau}_i \leq t} = \begin{cases} \sum_i \xi_i 1_{\tau_i \leq t} & \text{if } t \leq \tau, \\ \sum_i \xi_i 1_{\tau_i \leq \tau} + \sum_{i,j} 1_{A_j} (\xi_i^j 1_{\tau_i^j \leq t}) (\Pi_2^{t_j}(W, N)) & \text{if } t > \tau. \end{cases}$$

In other word, once  $\tau$  is reached, the impulse will be modified so that there would be no impulses on  $[\tau, \hat{\tau})$ , and starting at  $\hat{\tau}$ , the impulse follows the rule  $(\tau_i^j, \xi_i^j)$  on the set  $A_j$ . Now we have,

$$\begin{aligned} V(t_0, x_0) &\leq J[t_0, x_0, \hat{\tau}_i, \hat{\xi}_i] \\ &= \mathbb{E} \left[ \int_{t_0}^{\hat{\tau}} f(s, X_s^{t_0, x_0, \hat{\tau}_i, \hat{\xi}_i}) ds + \sum_i B(\hat{\tau}_i, \hat{\xi}_i) 1_{\hat{\tau}_i < \hat{\tau}} + J[\hat{\tau}, X_{\hat{\tau}-}^{t_0, x_0, \hat{\tau}_i, \hat{\xi}_i}, \hat{\tau}_i^\omega, \hat{\xi}_i^\omega] \right] \\ &= \mathbb{E} \left[ \int_{t_0}^{\tau} f(s, X_s^{t_0, x_0, \tau_i, \xi_i}) ds \right] + \mathbb{E} \left[ \int_{\tau}^{\hat{\tau}} f(s, X_s^{t_0, x_0, \hat{\tau}_i, \hat{\xi}_i}) ds \right] + \mathbb{E} \left[ \sum_i B(\tau_i, \xi_i) 1_{\tau_i < \tau} \right] \\ &\quad + \mathbb{E} \left[ \sum_j J[t_j, X_{t_j-}^{t_0, x_0, \tau_i, \xi_i}, \hat{\tau}_i^\omega, \hat{\xi}_i^\omega] 1_{A_j} \right]. \end{aligned}$$

The last equality follows from the fact that,  $\hat{\tau}_i$  is either  $< \tau$ , or  $\geq \hat{\tau}$ , so  $\hat{\tau}_i < \hat{\tau}$  implies that  $\hat{\tau}_i = \tau_i < \tau$ . Since  $\hat{u}^\omega = u.(\Pi^{-1}(\Pi_1^\dagger(W(\omega), N(\omega)), \Pi_2(\cdot))) = u^j(\cdot)$  on the set  $A_j$ ,

$$\begin{aligned} V(t_0, x_0) &\leq \mathbb{E} \left[ \int_{t_0}^{\tau} f(s, X_s^{t_0, x_0, \tau_i, \xi_i}, u_s) ds \right] + \mathbb{E} \left[ \int_{\tau}^{\hat{\tau}} f(s, X_s^{t_0, x_0, \hat{\tau}_i, \hat{\xi}_i}) ds \right] \\ &\quad + \mathbb{E} \left[ \sum_i B(\tau_i, \xi_i) 1_{\tau_i \leq \tau} \right] + \mathbb{E} \left[ \sum_j J \left[ t_j, X_{t_j-}^{t_0, x_0, \hat{\tau}_i, \hat{\xi}_i}, \tau_i^j, \xi_i^j \right] 1_{A_j} \right] \\ &\leq \mathbb{E} \left[ \int_{t_0}^{\tau} f(s, X_s^{t_0, x_0, \tau_i, \xi_i}) ds \right] + \mathbb{E} \left[ \int_{\tau}^{\hat{\tau}} f(s, X_s^{t_0, x_0, \hat{\tau}_i, \hat{\xi}_i}) ds \right] \\ &\quad + \mathbb{E} \left[ \sum_i B(\tau_i, \xi_i) 1_{\tau_i \leq \tau} \right] + \mathbb{E} \left[ \sum_j V(t_j, X_{t_j-}^{t_0, x_0, \hat{\tau}_i, \hat{\xi}_i}) 1_{A_j} \right] + \epsilon. \end{aligned}$$

Now, for the second term in the last expression, we see

$$\begin{aligned} &\mathbb{E} \left[ \int_{\tau}^{\hat{\tau}} f(s, X_s^{t_0, x_0, \hat{\tau}_i, \hat{\xi}_i}) ds \right] \leq \mathbb{E} \left[ \int_{\tau}^{\hat{\tau}} C(1 + |X_s|^{\gamma+\delta}) ds \right] \\ &\leq \mathbb{E} \left[ \int_{\tau}^{\hat{\tau}} C(1 + \mathbb{E}^\omega |X_s - X_\tau|^{\gamma+\delta} + |X_\tau|^{\gamma+\delta}) ds \right] \\ &\leq \mathbb{E} \left[ \int_{\tau}^{\hat{\tau}} C(1 + |X_\tau|^\mu) ds \right] \leq \mathbb{E} \left[ \mathbb{E}^\omega \left[ \int_{\tau}^{\hat{\tau}} C(1 + |X_\tau|^\mu) ds \right] \right] \\ &\leq \mathbb{E} [\epsilon C(1 + |X_\tau|^\mu)] \leq C\epsilon(1 + |x_0|^\mu). \end{aligned} \tag{12}$$

Therefore, it suffices to bound the following expression,

$$\mathbb{E} \left[ \sum_j [V(t_j, X_{t_j-}^{t_0, x_0, \hat{u}, \hat{\tau}_i, \hat{\xi}_i}) - V(\tau, X_{\tau-}^{t_0, x_0, u, \tau_i, \xi_i})] 1_{A_j} \right].$$

First, note that on the interval  $[\tau, \hat{\tau})$ ,  $X = X^{t_0, x_0, \hat{u}, \hat{\tau}_i, \hat{\xi}_i}$  solves the jump SDE with no impulse:

$$\begin{aligned} X_{t \wedge \tau} - X_\tau &= \int_{\tau}^{t \wedge \tau} b(s, X_s) dt + \int_{\tau}^{t \wedge \tau} \sigma(s, X_s) dW \\ &\quad + \int_{\tau}^{t \wedge \tau} \int_{0 < |j| < 1} j(s, X_s, z) \tilde{N}(dz, ds) \\ &\quad + \int_{\tau}^{t \wedge \tau} \int_{1 \leq |j|} j(s, X_s, z) N(dz, ds). \end{aligned}$$

In particular, under  $(\mathbb{P}|(\mathcal{F}^\circ)_{t_0, \tau}^{W, N})$ ,  $\tau$ ,  $X_\tau^{t_0, x_0, \hat{\tau}_i, \hat{\xi}_i}$  and  $\hat{\tau}$  are all deterministic, hence the following estimates

$$\begin{aligned} \mathbb{E}^{(\mathbb{P}|(\mathcal{F}^\circ)_{t_0, \tau}^{W, N})(\omega)} |X_{\hat{\tau}(\omega)}|^\beta &\leq C(1 + |X_{\tau(\omega)}|^\beta), & \text{if } \beta > 0, \\ \mathbb{E}^{(\mathbb{P}|(\mathcal{F}^\circ)_{t_0, \tau}^{W, N})(\omega)} |X_{\hat{\tau}(\omega)} - X_{\tau(\omega)}|^\beta &\leq C(1 + |X_{\tau(\omega)}|^\beta)(\hat{\tau}(\omega) - \tau(\omega))^{\beta/2 \wedge 1} \\ &\leq C(1 + |X_{\tau(\omega)}|^\beta) \epsilon^{\beta/2 \wedge 1}, & \text{if } \beta \geq \nu. \end{aligned}$$

Thus, let  $E^\omega = \mathbb{E}(\mathbb{P}^{(F^o)_{t_0, \tau}^{W, N}}(\omega))$ , we see

$$\begin{aligned} & \mathbb{E}^\omega [V(t_j, X_{t_j-}^{t_0, x_0, \hat{\tau}_i, \hat{\xi}_i}) - V(\tau, X_{\tau-}^{t_0, x_0, \tau_i, \xi_i})] \\ & \leq \mathbb{E}^\omega \left[ C(1 + |X_\tau(\omega)|^\mu + |X_{\hat{\tau}}|^\mu) |\hat{\tau} - \tau(\omega)|^{\delta/2} + C(1 + |X_\tau(\omega)|^\gamma + |X_{\hat{\tau}}|^\gamma) |X_{\hat{\tau}} - X_\tau(\omega)|^\delta \right] \\ & \leq C(1 + |X_\tau(\omega)|^\mu + \mathbb{E}^\omega |X_{\hat{\tau}}|^\mu) \epsilon^{\delta/2} + C \mathbb{E}^\omega [(1 + |X_\tau(\omega)|^\gamma + |X_{\hat{\tau}}|^\gamma) |X_{\hat{\tau}} - X_\tau(\omega)|^\delta] \\ & \leq C(1 + |X_\tau(\omega)|^\mu) \epsilon^{\delta/2} + C \left( \mathbb{E}^\omega [(1 + |X_\tau(\omega)|^\gamma + |X_{\hat{\tau}}|^\gamma)^{p'}] \right)^{1/p'} \left( \mathbb{E}^\omega [|X_{\hat{\tau}} - X_\tau(\omega)|^\delta]^p \right)^{1/p} \end{aligned}$$

(where  $p = \mu/\delta > 0$ , and  $1/p + 1/p' = 1$ )

$$\begin{aligned} & \leq C(1 + |X_\tau(\omega)|^\mu) \epsilon^{\delta/2} + C \left( 1 + |X_\tau(\omega)|^\gamma + \left( \mathbb{E}^\omega |X_{\hat{\tau}}|^{p'} \right)^{1/p'} \right) (\mathbb{E}^\omega |X_{\hat{\tau}} - X_\tau(\omega)|^\mu)^{\delta/\mu} \\ & \leq C(1 + |X_\tau(\omega)|^\mu) \epsilon^{\delta/2} + C(1 + |X_\tau(\omega)|^\gamma) (1 + |X_{\tau(\omega)}|^\mu)^{\delta/\mu} \epsilon^{(\nu/2 \wedge 1) \frac{\delta}{\mu}} \\ & \leq C(1 + |X_\tau(\omega)|^\mu) \epsilon^{\delta/2} + C(1 + |X_\tau(\omega)|^\mu) \epsilon^{(\nu/2 \wedge 1) \frac{\delta}{\mu}}. \end{aligned}$$

Taking expectation, we get

$$\begin{aligned} & \mathbb{E} \left[ \sum_j [V(t_j, X_{t_j-}^{t_0, x_0, \hat{\tau}_i, \hat{\xi}_i}) - V(\tau, X_{\tau-}^{t_0, x_0, \tau_i, \xi_i})] 1_{A_j} \right] \\ & \leq C(1 + \mathbb{E} |X_\tau|^\mu) \epsilon^{\delta/2} + C(1 + \mathbb{E} |X_\tau|^\mu) \epsilon^{(\nu/2 \wedge 1) \frac{\delta}{\mu}} \\ & \leq C(1 + |x_0|^\mu) \epsilon^{\delta/2} + C(1 + |x_0|^\mu) \epsilon^{(\nu/2 \wedge 1) \frac{\delta}{\mu}}. \end{aligned}$$

The last inequality follows from Corollary 3.7 in Tang & Yong [34].

With these two bounds, and taking  $\epsilon \rightarrow 0$ , we get the desired inequality and the DPP.  $\square$

### 3.2 Preliminary Results

To analyze the value function, we also need some preliminary results, in addition to the DPP.

**Lemma 4.** *The set*

$$\Xi(x, t) := \{\xi \in \mathbb{R}^n : MV(x, t) = V(x + \xi, t) + B(\xi, t)\}$$

*is nonempty, i.e. the infimum is in fact a minimum. Moreover, for  $(x, t)$  in bounded  $B' \subset \mathbb{R}^n \times [0, T]$ ,  $\{(y, t) : y = x + \xi, (x, t) \in B', \xi \in \Xi(x, t)\}$  is also bounded.*

*Proof.* This is easy by  $B(\xi, t) \geq L + C|\xi|^\mu$ ,  $-C \leq V \leq C(1 + |x|^{\gamma+\delta})$ , and  $\mu > \gamma + \delta$ .  $\square$

**Lemma 5.** (Theorem 4.9 in [24]) *Assume that  $a_{ij}, b_i, f \in C_{loc}^\alpha(\mathbb{R}^n \times (0, T))$ . If  $-u_t + Lu = f$  in  $\mathcal{C}$  in the viscosity sense, then it solves the PDE in the classical sense as well, and  $u(x, T-t) \in C_{loc}^{2+\alpha, 1+\frac{\alpha}{2}}(\mathcal{C})$ .*

**Lemma 6.** *The value function  $V$  and  $MV$  satisfies  $V(x, t) \leq MV(x, t)$  pointwise.*

**Lemma 7.**  *$MV$  is continuous, and there exists  $C$  such that for any  $x, y \in \mathbb{R}^n$ ,  $s < t$ ,*

$$\begin{aligned} |MV(x, t) - MV(y, t)| & \leq C(1 + |x|^\gamma + |y|^\gamma) |x - y|^\delta, \\ MV(x, t) - MV(x, s) & \leq C(1 + |x|^\mu) |t - s|^{\delta/2}. \end{aligned}$$

*Proof.* First we prove continuity. For each  $\xi$ ,  $V(x, t) + B(\xi, t)$  is a uniformly continuous function on compact sets. And since  $\Xi(x, t)$  is bounded for  $(x, t)$  on compact sets, taking the infimum over  $\xi$  on some fixed compact sets implies that  $MV$  is continuous.

For the Hölder continuity in  $t$ , let  $\xi \in \Xi(x, s)$ , then

$$\begin{aligned} & MV(x, t) - MV(x, s) \\ & \leq V(x + \xi, t) + B(\xi, t) - V(x + \xi, s) - B(\xi, s) \\ & \leq C(1 + |x|^\mu) |t - s|^{\delta/2}, \end{aligned}$$

given that  $B(\xi, s) \geq B(\xi, t)$  for  $s < t$ .  $\square$

As a consequence, the continuous region  $\mathcal{C}$  is open.

**Lemma 8.** Fix  $x$  in some bounded  $B \subset \mathbb{R}^n$ . Let  $\epsilon > 0$ . For any

$$\xi \in \Xi^\epsilon(x, t) = \{\xi : V(x + \xi, t) + B(\xi, t) < MV(x, t) + \epsilon\},$$

we have

$$V(x + \xi, t) + K - \epsilon < MV(x + \xi, t). \quad (11)$$

In particular, Let  $\mathcal{C}^{K/2} = \{(x, t) \in \mathbb{R}^n \times [0, T] : V(x, t) < MV(x, t) - K/2\}$ . Then if  $\xi \in \Xi^{K/2}(x, t)$ , then  $(x + \xi, t) \in \mathcal{C}^{K/2}$ .

*Proof.* Suppose  $\xi \in \Xi^\epsilon(x, t)$ , i.e.

$$V(x + \xi, t) + B(\xi, t) < MV(x, t) + \epsilon.$$

Then,

$$\begin{aligned} MV(x + \xi, t) &= \inf_{\eta} V(x + \xi + \eta, t) + B(\eta, t) \\ &= \inf_{\eta} V(x + \xi + \eta, t) + B(\xi + \eta, t) - B(\xi + \eta, t) + B(\eta, t) \\ &\geq \inf_{\eta} V(x + \xi + \eta, t) + B(\xi + \eta, t) - B(\xi, t) + K \\ &= \inf_{\eta'} V(x + \eta', t) + B(\eta', t) - B(\xi, t) + K \\ &= MV(x, t) - B(\xi, t) + K \\ &> V(x + \xi, t) - \epsilon + K. \end{aligned}$$

Let  $\epsilon = K/2$ , we get that  $\xi \in \Xi^{K/2}(x, t)$  implies  $x + \xi \in \mathcal{C}^{K/2}$ .  $\square$

**Lemma 9.**  $MV$  is uniformly semi-concave in  $x$ , and  $MV_t$  is bounded above in the distributional sense on compact sets away from  $t = T$ .

*Proof.* Let  $A$  be a compact subset of  $\mathbb{R}^n \times [0, T - \delta]$ . For any  $\xi \in \Xi(x, t)$  for  $(x, t) \in A$ ,  $(x + \xi, t)$  lies in a bounded region  $B$  independent of  $(x, t)$ . For any  $|y| = 1$  and  $\delta > 0$  sufficiently small,

$$\begin{aligned} &\frac{MV(x + \delta y, t) - 2MV(x, t) + MV(x - \delta y, t)}{2\delta} \\ &\leq \frac{(V(x + \delta y + \xi, t) + B(\xi, t)) - 2(V(x + \xi, t) + B(\xi, t)) + (V(x - \delta y + \xi, t) + B(\xi, t))}{2\delta} \\ &= \frac{V(x + \delta y + \xi, t) - 2V(x + \xi, t) + V(x - \delta y + \xi, t)}{2\delta} \\ &\leq C \|D^2 V\|_{B \cap \mathcal{C}^{K/2}}, \end{aligned}$$

which is bounded by Lemma 5. Similarly,

$$\begin{aligned} &\frac{MV(x, t + \delta) - MV(x, t)}{\delta} \\ &\leq \frac{V(x + \xi, t + \delta) + B(\xi, t + \delta) - (V(x + \xi, t) + B(\xi, t))}{\delta} \\ &= \frac{V(x + \xi, t + \delta) - V(x + \xi, t)}{\delta} + \frac{B(\xi, t + \delta) - B(\xi, t)}{\delta} \\ &\leq C \|V_t\|_{B \cap \mathcal{C}^{K/2}}. \end{aligned}$$

$\square$

## 4 Value function as a Viscosity Solution

In this section, we establish the value function  $V(x, t)$  as a viscosity solution to the (HJB) equation in the sense of [1].

**Theorem 2. (Viscosity Solution of the Value Function)** *The value function  $V(x, t)$  is a continuous viscosity solution to the (HJB) equation in the following sense: if for any  $\phi \in C^2(\mathbb{R}^n \times [0, T])$ ,*

1.  *$u - \phi$  achieves a local maximum at  $(x_0, t_0) \in B(x_0, \theta) \times [t_0, t_0 + \theta]$  with  $u(x_0, t_0) = \phi(x_0, t_0)$ , then  $V$  is a subsolution*

$$\max\{-\phi_t + L\phi - f - I_\theta^1[\phi] - I_\theta^2[u], u - Mu\}(x_0, t_0) \leq 0.$$

2.  *$u \geq \phi$  and  $u - \phi$  achieves a local minimum at  $(x_0, t_0) \in B(x_0, \theta) \times [t_0, t_0 + \theta]$  with  $u(x_0, t_0) = \phi(x_0, t_0)$ , then  $V$  is a supersolution*

$$\max\{-\phi_t + L\phi - f - I_\theta^1[\phi] - I_\theta^2[u], u - Mu\}(x_0, t_0) \geq 0.$$

Here

$$\begin{aligned} I_\theta^1[\phi](x, t) &= \int_{|z| < \theta} \phi(x + z, t) - \phi(x, t) - D\phi(x, t) \cdot z 1_{|z| < 1} \rho(dz), \\ I_\theta^2[u](x, t) &= \int_{|z| \geq \theta} u(x + z, t) - u(x, t) - D\phi(x, t) \cdot z 1_{|z| < 1} \rho(dz), \end{aligned}$$

with the boundary condition  $u = g$  on  $\mathbb{R}^n \times \{t = T\}$ .

*Proof.* First, suppose  $V - \phi$  achieves a local maximum in  $B(x_0, \theta) \times [t_0, t_0 + \theta]$  with  $V(x_0, t_0) = \phi(x_0, t_0)$ , we prove by contradiction that  $(-\phi_t + L\phi - I_\theta^1[\phi] - I_\theta^2[V] - f)(x_0, t_0) \leq 0$ .

Suppose otherwise, i.e.  $(-\phi_t + L\phi - I_\theta^1[\phi] - I_\theta^2[V] - f)(x_0, t_0) > 0$ . Then without loss of generality we can assume that  $-\phi_t + L\phi - I_\theta^1[\phi] - I_\theta^2[V] - f > 0$  on  $B(x_0, \theta) \times [t_0, t_0 + \theta]$ . Since the definition of viscosity solution does not concern the value of  $\phi$  outside of  $B(x_0, \theta) \times [t_0, t_0 + \theta]$ , we can assume that  $\phi$  is bounded by multiples of  $|V|$ . Let  $X^0 = X^{x_0, t_0, \infty, 0}$  and

$$\tau = \inf\{t \in [t_0, T] : X_t \notin B(x_0, \theta) \times [t_0, t_0 + \theta]\} \wedge T.$$

By Ito's formula,

$$\mathbb{E}[\phi(X_\tau^0, \tau)] - \phi(x_0, t_0) = \mathbb{E}\left[\int_{t_0}^\tau (\phi_t - L\phi + I_\theta^1[\phi] + I_\theta^2[\phi])(X_{s-}^0, s) ds\right].$$

Meanwhile, by Theorem 1,

$$V(x_0, t_0) \leq \mathbb{E}\left[\int_{t_0}^\tau f(X_s^0, s) ds + V(X_\tau^0, \tau)\right].$$

Combining these two inequalities, we get

$$\begin{aligned} & \mathbb{E}[V(X_\tau^0, \tau)] - \mathbb{E}\left[\int_{t_0}^\tau (\phi_t - L\phi + I_\theta^1[\phi] + I_\theta^2[\phi])(X_{s-}^0, s) ds\right] \\ & \leq \mathbb{E}[\phi(X_\tau^0, \tau)] - \mathbb{E}\left[\int_{t_0}^\tau (\phi_t - L\phi + I_\theta^1[\phi] + I_\theta^2[\phi])(X_{s-}^0, s) ds\right] \\ & = \phi(x_0, t_0) = V(x_0, t_0) \leq \mathbb{E}\left[\int_{t_0}^\tau f(X_s^0, s) ds + V(X_\tau^0, \tau)\right]. \end{aligned}$$

That is,

$$\mathbb{E}\left[\int_{t_0}^\tau (-\phi_t + L\phi - I_\theta^1[\phi] - I_\theta^2[\phi] - f)(X_{s-}^0, s) ds\right] \leq 0.$$

Again by modifying the value of  $\phi$  outside of  $B(x_0, \theta) \times [t_0, t_0 + \theta)$ , and since  $V \leq \phi$  in  $B(x_0, \theta) \times [t_0, t_0 + \theta)$ , we can take a sequence of  $\phi_k \geq V$  dominated by multiples of  $|V|$  such that it converges to  $V$  outside of  $B(x_0, \theta) \times [t_0, t_0 + \theta)$  from above. By the dominated convergence theorem,  $I_\theta^2[\phi]$  converges to  $I_\theta^2[V]$ . Thus,

$$\mathbb{E} \left[ \int_{t_0}^{\tau} (-\phi_t + L\phi - I_\theta^1[\phi] - I_\theta^2[V] - f)(X_{s-}^0, s) ds \right] \leq 0,$$

which is a contradiction. Therefore, we must have  $(-\phi_t + L\phi - I_\theta^1[\phi] - I_\theta^2[V] - f)(x_0, t_0) \leq 0$ , and since  $V \leq MV$ , we have  $\max\{-\phi_t + L\phi - I_\theta^1[\phi] - I_\theta^2[V] - f, V - MV\}(x_0, t_0) \leq 0$ .

Next, suppose  $V - \phi$  achieves local minimum in  $B(x_0, \theta) \times [t_0, t_0 + \theta)$  with  $V(x_0, t_0) = \phi(x_0, t_0)$ . Then if  $(V - MV)(x_0, t_0) = 0$ , then we already have the desired inequality. Now suppose  $V - MV \leq -\epsilon < 0$  and  $-\phi_t + L\phi - I_\theta^1[\phi] - I_\theta^2[V] - f < 0$  on  $B(x_0, \theta) \times [t_0, t_0 + \theta)$ . Assuming as before that  $\phi$  is bounded by multiples of  $|V|$  outside of  $B(x_0, \theta) \times [t_0, t_0 + \theta)$ . By Ito's formula

$$\mathbb{E} [\phi(X_\tau^0, \tau)] - \phi(x_0, t_0) = \mathbb{E} \left[ \int_{t_0}^{\tau} (\phi_t - L\phi + I_\theta^1[\phi] + I_\theta^2[V])(X_{s-}^0, s) ds \right].$$

Consider the no impulse strategy  $\tau_i^* = \infty$  and let  $X^0 = X^{t_0, x_0, \infty, 0}$ . Define the stopping time  $\tau$  as before, i.e.,

$$\tau = \inf\{t \in [t_0, T] : X_t \notin B(x_0, \theta) \times [t_0, t_0 + \theta)\} \wedge T.$$

Then for any strategy  $(\tau_i, \xi_i) \in \mathcal{V}$ ,

$$\begin{aligned} & J[t_0, x_0, \tau_i, \xi_i] \\ &= \mathbb{E} \left[ \int_{t_0}^{\tau_1 \wedge \tau} f(s, X_s^{t_0, x_0, \tau_i, \xi_i}) ds + B(\tau_1, \xi_1) 1_{\{\tau_1 \leq \tau \wedge \tau_1\}} + J[\tau_1 \wedge \tau, X_{\tau_1 \wedge \tau}^{t_0, x_0, \tau_i, \xi_i}, \tau_i, \xi_i] \right] \\ &\geq \mathbb{E} \left[ \int_{t_0}^{\tau_1 \wedge \tau} f(s, X_s^{t_0, x_0, \tau_i, \xi_i}) ds + 1_{\{\tau_1 \leq \tau\}} (B(\tau_1, \xi_1) + V(X_{\tau_1}^{t_0, x_0, \tau_i, \xi_i}, \tau_1)) \right] \\ &\quad + \mathbb{E} [1_{\{\tau_1 > \tau\}} V(X_\tau^{t_0, x_0, \tau_i, \xi_i}, \tau)] \\ &\geq \mathbb{E} \left[ \int_{t_0}^{\tau_1 \wedge \tau} f(s, X_s^0) ds + 1_{\{\tau_1 \leq \tau\}} MV(X_{\tau_1}^0, \tau_1) \right] \\ &\quad + \mathbb{E} [1_{\{\tau_1 > \tau\}} V(X_\tau^0, \tau)] \\ &\geq \mathbb{E} \left[ \int_{t_0}^{\tau_1 \wedge \tau} f(s, X_s^0) ds + V(X_{\tau_1 \wedge \tau}^0, \tau_1 \wedge \tau) \right] + \epsilon \cdot \mathbb{P}(\tau_1 \leq \tau) \\ &\geq V(t_0, x_0) + \epsilon \cdot \mathbb{P}(\tau_1 \leq \tau). \end{aligned}$$

Therefore, without loss of generality, we only need to consider  $(\tau_i, \xi_i) \in \mathcal{V}$  such that  $\tau_1 > \tau$ . Now, the Dynamic Programming Principle becomes,

$$u(x_0, t_0) = \mathbb{E} \left[ \int_{t_0}^{\tau} f(X_s^0, s) ds + V(X_\tau^0, \tau) \right].$$

Now combining these facts above,

$$\begin{aligned} & \mathbb{E} [V(X_\tau^0, \tau)] - \mathbb{E} \left[ \int_{t_0}^{\tau} (\phi_t - L\phi + I_\theta^1[\phi] + I_\theta^2[\phi])(X_{s-}^0, s) ds \right] \\ &\geq \mathbb{E} [\phi(X_\tau^0, \tau)] - \mathbb{E} \left[ \int_{t_0}^{\tau} (\phi_t - L\phi + I_\theta^1[\phi] + I_\theta^2[\phi])(X_{s-}^0, s) ds \right] \\ &= \phi(x_0, t_0) = V(x_0, t_0) = \mathbb{E} \left[ \int_{t_0}^{\tau} f(X_s^0, s) ds + V(X_\tau^0, \tau) \right] \\ &\quad - \mathbb{E} \left[ \int_{t_0}^{\tau} (-\phi_t + L\phi - I_\theta^1[\phi] - I_\theta^2[\phi] - f)(X_{s-}^0, s) ds \right] \geq 0. \end{aligned}$$

Again by modifying the value of  $\phi$  outside of  $B(x_0, \theta) \times [t_0, t_0 + \theta)$ , and since  $V \geq \phi$  in  $B(x_0, \theta) \times [t_0, t_0 + \theta)$ , we can take a sequence of  $\phi_k \leq u$  dominated by multiples of  $|V|$  such that it converges to  $V$  outside of  $B(x_0, \theta) \times [t_0, t_0 + \theta)$  from above. By the dominated convergence theorem,  $I_\theta^2[\phi]$  converges to  $I_\theta^2[V]$ . Hence

$$\mathbb{E} \left[ \int_{t_0}^{\tau} (-\phi_t + L\phi - I_\theta^1[\phi] - I_\theta^2[V] - f)(X_{s-}^0, s) ds \right] \geq 0,$$

which contradicts the assumption that  $(-\phi_t + L\phi - I_\theta^1[\phi] - I_\theta^2[V] - f)(x_0, t_0) < 0$ . Therefore,  $\max\{-\phi_t + L\phi - I_\theta^1[\phi] - I_\theta^2[V] - f, V - MV\}(x_0, t_0) \geq 0$ .  $\square$

## 5 Regularity of the Value Function

To study the regularity of the value function, we will consider the time-inverted value function  $u(x, t) = V(x, T - t)$ . Accordingly, we will assume that  $a_{ij}$ ,  $b_i$ ,  $f$ ,  $B$  and  $j$  are all time-inverted. This is to be consistent with the standard PDE literature for easy references to some of its classical results, where the value is specified at the initial time instead of the terminal time.

The regularity study is built in two phases.

First in Section 5.1, we focus on the case without jumps. We will construct a unique  $W_{loc}^{(2,1),p}$  regular viscosity solution to a corresponding equation without the integro-differential operator on a fixed bounded domain  $Q_T$  with  $Q_T = B(0, R) \times (\delta, T]$  for  $R > 0$  and  $\delta > 0$ ,

$$\begin{cases} \max\{u_t + Lu - f, u - \Psi\} = 0 & \text{in } Q_T, \\ u(t, x) = \phi & \text{on } \partial_P Q_T. \end{cases} \quad (12)$$

in which  $\phi(x, t) = V(x, T - t)$  and  $\Psi(x, t) = (Mu)(x, t)$ . The local uniqueness of the viscosity solution then implies that this solution must be the time-inverted value function, hence the  $W_{loc}^{(2,1),p}$  smoothness for the value function.

Then in Section 5.2, we extend the analysis to the case with a first-order jump and establish the regularity property of the value function.

### 5.1 $W_{loc}^{(2,1),p}$ Regularity for cases without jumps

The key idea is to study a corresponding homogenous HJB, based on the following classical result in PDEs.

**Lemma 10.** (Theorem 4.9, 5.9, 5.10, and 6.33 of [24]) Let  $\alpha \in (0, 1]$ . Assume that  $a_{ij}, b_i, f \in C^{0+\alpha, 0+\frac{\alpha}{2}}(\overline{Q_T})$ ,  $a_{ij}$  is uniformly elliptic,  $\phi \in C^{0+\alpha, 0+\frac{\alpha}{2}}(\partial_P Q_T)$ . Then the linear PDE

$$\begin{cases} u_t + Lu = f & \text{in } Q_T; \\ u = \phi & \text{on } \partial_P Q_T. \end{cases} \quad (13)$$

has a unique solution to (13) that lies in  $C^{0+\alpha, 0+\frac{\alpha}{2}}(\overline{Q_T}) \cap C_{loc}^{2+\alpha, 1+\frac{\alpha}{2}}(Q_T)$ .

Indeed, given Lemma 10, let  $u_0$  be the unique classical solution to (13), with the boundary condition  $\phi(x, t) = V(T - t, x)$ . Then, our earlier analysis (Lemma 3) of Hölder continuity for the value function implies that  $V(x, T - t) - u_0(x, t)$  solves the following ‘‘homogenous’’ HJB,

$$\begin{cases} \max\{u_t + Lu, u - \bar{\Psi}\} = 0 & \text{in } Q_T, \\ u = 0 & \text{on } \partial_P Q_T. \end{cases}$$

for  $\bar{\Psi} = \Psi - u_0$ . Since  $V \leq MV$ , we have  $\bar{\Psi}(x, t) = (\Psi - u_0)(x, t) = (MV - V)(x, T - t) \geq 0$  on  $\partial_P Q_T$ .

Therefore, our first step is to study the above ‘‘homogenous’’ HJB.

## Step I: Viscosity solution of the “homogenous” HJB

**Theorem 3.** *Assume*

1.  $a_{ij}, b_i, \bar{\Psi} \in C^{0+\alpha, 0+\frac{\alpha}{2}}(\bar{Q}_T)$ ,
2.  $(a_{ij})$  uniformly elliptic,
3.  $\bar{\Psi}$  is semiconcave,
4.  $\bar{\Psi}_t$  is bounded below, in the distributional sense.

Then there exists a viscosity solution  $u \in W^{(2,1),p}(Q_T)$  to the homogenous HJB

$$\begin{cases} \max\{u_t + Lu, u - \bar{\Psi}\} = 0 & \text{in } Q_T, \\ u = 0 & \text{on } \partial_P Q_T. \end{cases} \quad (14)$$

In fact,  $u \in W^{(2,1),p}(Q_T)$  for any  $p > 1$ .

To prove this theorem, we first consider a corresponding penalized version. For every  $\epsilon > 0$ , let  $\beta_\epsilon : \mathbb{R} \rightarrow \mathbb{R}$  be a smooth function such that  $\beta_\epsilon(x) \geq -1$ ,  $\beta_\epsilon(0) = 0$ ,  $\beta'_\epsilon > 0$ ,  $\beta''_\epsilon \geq 0$ ,  $\beta'_\epsilon(x) \leq C/\epsilon$  for  $x \geq 0$ ,  $\beta'_\epsilon(0) = 1/\epsilon$  and as  $\epsilon \rightarrow 0$ ,  $\beta_\epsilon(x) \rightarrow \infty$  for  $x > 0$ ,  $\beta_\epsilon(x) \rightarrow 0$  for  $x < 0$ . One such example is,  $\beta(x) = x/\epsilon$  for  $x \geq 0$  and its smooth extension to  $x < 0$ . We see that there is a classical solution  $u$  to the penalized problem, assuming some regularity on the coefficients  $a^{ij}, b^i, \bar{\Psi}$ .

**Lemma 11.** *Fix  $\epsilon > 0$ . Suppose that  $a^{ij}, b^i, \bar{\Psi} \in C^{2+\alpha, 1+\alpha/2}(\bar{Q}_T)$ , and  $(a^{ij})$  is uniformly elliptic. Then exists a unique  $u \in C^{4+\alpha, 2+\alpha/2}(\bar{Q}_T)$  such that*

$$\begin{cases} u_t + Lu + \beta_\epsilon(u - \bar{\Psi}) = 0 & \text{on } Q_T, \\ u = 0 & \text{on } \partial_P Q_T. \end{cases} \quad (15)$$

Note that Friedman [16] proved a similar result for a  $W^{2,p}$  solution for the elliptic case using the  $L^p$  estimates. He then used the Hölder estimates to bootstrap for the  $C^2$  regularity. Our proof is more elementary using only the Schauder estimates. (For details, see Appendix B).

Next, consider the case with  $C^{\alpha, \alpha/2}(\bar{Q}_T)$  coefficients. We will smooth out the coefficients first to the above result, and then let  $\epsilon \rightarrow 0$ . More precisely, let  $(a^\epsilon)^{ij}, (b^\epsilon)^i, \bar{\Psi}^\epsilon \in C^\infty(\bar{Q}_T)$  be such that they converge to the respective function in  $C^{\alpha, \alpha/2}(\bar{Q}_T)$  and  $\bar{\Psi}^\epsilon \geq 0$  on  $\partial_P Q_T$ . This is possible because  $\bar{\Psi} \geq 0$  on  $\partial_P Q_T$ . Define  $L^\epsilon$  to be the corresponding linear operator and  $u^\epsilon$  to be the unique solution to

$$\begin{cases} u_t^\epsilon + L^\epsilon u^\epsilon + \beta_\epsilon(u^\epsilon - \bar{\Psi}^\epsilon) = 0 & \text{on } Q_T, \\ u^\epsilon = 0 & \text{on } \partial_P Q_T. \end{cases} \quad (16)$$

Now we establish some bound for  $\beta_\epsilon(u^\epsilon - \bar{\Psi}^\epsilon)$ , in order to apply an  $L^p$  estimate.

**Lemma 12.** *Assuming  $\bar{\Psi}$  is semiconcave in  $x$ , i.e.*

$$\frac{\partial^2 \bar{\Psi}}{\partial \xi^2} \leq C, \quad (17)$$

for any direction  $|\xi| = 1$ , and

$$\frac{\partial \bar{\Psi}}{\partial t} \geq -C, \quad (18)$$

where both derivatives are interpreted in the distributional sense. We have

$$|\beta_\epsilon(u^\epsilon - \bar{\Psi}^\epsilon)| \leq C, \quad (19)$$

with  $C$  independent of  $\epsilon$ .



*Proof.* Clearly  $\beta_\epsilon \geq -1$ , so we only need to give an upper bound. The assumption above translates to the same derivative condition on mollified  $\overline{\Psi}^\epsilon$ , which can be interpreted classically now. Thus we have

$$\overline{\Psi}_t^\epsilon + L^\epsilon \overline{\Psi}^\epsilon \geq -C. \quad (20)$$

Suppose  $u^\epsilon - \overline{\Psi}^\epsilon$  achieves maximum at  $(x_0, t_0) \in Q_T$ , then

$$(u^\epsilon - \overline{\Psi}^\epsilon)_t + L^\epsilon(u^\epsilon - \overline{\Psi}^\epsilon)(x_0, t_0) \geq 0. \quad (21)$$

Hence

$$-\beta_\epsilon(u^\epsilon - \overline{\Psi}^\epsilon)(x_0, t_0) = (u_t^\epsilon + L^\epsilon u^\epsilon)(x_0, t_0) \quad (22)$$

$$\geq (\overline{\Psi}_t^\epsilon + L^\epsilon \overline{\Psi}^\epsilon)(x_0, t_0) \geq -C, \quad (23)$$

in which  $C$  is an upper bound independent of  $\epsilon$ . On the other hand, if it achieves maximum on  $\partial_P Q_T$ , we get  $u^\epsilon - \overline{\Psi}^\epsilon \leq 0$  since  $\overline{\Psi}^\epsilon \geq 0$  on  $\partial_P Q_T$ . Either way we have an upper bound independent of  $\epsilon$ .  $\square$

Now with this estimate of the boundedness of  $\beta_\epsilon(u^\epsilon - \overline{\Psi}^\epsilon)$ , we are ready to prove Theorem 3.

*Proof.* Lemma 12 allows us to apply  $L^p$  estimate:

$$\|u^\epsilon\|_{W^{(2,1),p}(Q_T)} \leq C \|\beta_\epsilon(u^\epsilon - \overline{\Psi}^\epsilon)\|_{L^p(Q_T)} \leq C, \quad (24)$$

for  $p > 1$ . Thus there exists a sequence  $\epsilon_n \rightarrow 0$  and  $u \in W^{(2,1),p}(Q_T)$  such that

$$u^{\epsilon_n} \rightharpoonup u \quad (25)$$

weakly in  $W^{(2,1),p}(Q_T)$ . For  $p$  large enough, there exists  $\alpha' > 0$  such that  $u^\epsilon \rightarrow u$  in  $C^{\alpha', \alpha'/2}(\overline{Q}_T)$ , so  $u^\epsilon \rightarrow u$  uniformly in  $\overline{Q}_T$ .

On one hand, since  $\beta_\epsilon(u^\epsilon - \overline{\Psi}^\epsilon) \leq C$ , yet  $\beta_\epsilon(x) \rightarrow \infty$  as  $x > 0$ , hence  $u \leq \overline{\Psi}$ . Suppose  $u - \phi$  achieves a strict local maximum at  $(x_0, t_0)$ , then  $u^\epsilon - \phi$  achieves a strict local maximum at  $(x_0^\epsilon, t_0^\epsilon)$  and  $(x_0^\epsilon, t_0^\epsilon) \rightarrow (x_0, t_0)$  as  $\epsilon \rightarrow 0$ , then

$$\lim_{\epsilon \rightarrow 0} (\phi_t + L^\epsilon \phi)(x_0^\epsilon, t_0^\epsilon) \leq \liminf_{\epsilon \rightarrow 0} \beta_\epsilon(u^\epsilon(x_0^\epsilon, t_0^\epsilon) - \overline{\Psi}(x_0^\epsilon, t_0^\epsilon)) \leq 0.$$

So  $(\phi_t + L\phi)(x_0, t_0) \leq 0$ .

On the other hand, if  $u - \phi$  achieves a strict local minimum at  $(x_0, t_0)$ , then  $u^\epsilon - \phi$  achieves a strict local maximum at  $(x_0^\epsilon, t_0^\epsilon)$  and  $(x_0^\epsilon, t_0^\epsilon) \rightarrow (x_0, t_0)$  as  $\epsilon \rightarrow 0$ . If  $u(x_0, t_0) < \overline{\Psi}(x_0, t_0)$ , then for small  $\epsilon$ ,  $u(x_0^\epsilon, t_0^\epsilon) < \overline{\Psi}(x_0^\epsilon, t_0^\epsilon)$ ,

$$\lim_{\epsilon \rightarrow 0} (\phi_t + L^\epsilon \phi)(x_0^\epsilon, t_0^\epsilon) \geq \limsup_{\epsilon \rightarrow 0} \beta_\epsilon(u^\epsilon(x_0^\epsilon, t_0^\epsilon) - \overline{\Psi}(x_0^\epsilon, t_0^\epsilon)) \geq 0.$$

$\square$

## Step II: Uniqueness of the HJB equation without jump terms

**Proposition 2.** Assuming that  $a_{ij}, b_i, f, \Psi, f$  are continuous in  $\overline{Q}_T$ , and  $\phi$  continuous on  $\partial_P Q_T$ , the viscosity solution to the following HJB equation is unique.

$$\begin{cases} \max\{u_t + Lu - f, u - \Psi\} = 0 & \text{in } Q_T, \\ u(t, x) = \phi & \text{on } \partial_P Q_T. \end{cases} \quad (26)$$

*Remark.* Note that this is a local uniqueness of the viscosity solution. We later apply  $\phi(x, t) = V(x, T - t)$  and  $\Psi(x, t) = (Mu)(x, t)$  to our original control problem.

*Proof.* Let  $W, U$  be a viscosity subsolution and supersolution to (26) respectively. Then  $W$  is clearly a viscosity subsolution to  $v_t + Lv - f = 0$ , with  $W \leq \bar{\Psi}$ . On the other hand, at any fixed point  $(x_0, t_0)$ , either  $U(x_0, t_0) = \bar{\Psi}(x_0, t_0)$  or  $U$  satisfies the viscosity supersolution property at  $(x_0, t_0)$ .

Define

$$W^\epsilon(x, t) = W(x, t) + \frac{\epsilon}{t - \delta} \quad (27)$$

for  $\epsilon > 0$ . Note that  $W^\epsilon$  is still a viscosity subsolution of  $v_t + Lv - f = 0$ . For fixed  $\epsilon, \alpha, \beta$ , define

$$\Phi(t, x, y) = W^\epsilon(x, t) - U(x, t) - \alpha|x - y|^2 - \beta(t - \delta). \quad (28)$$

Denote  $B = B(0, R)$ . Suppose  $\max_{(x, t) \in \bar{Q}_T} W^\epsilon(x, t) - U(x, t) \geq c > 0$ . There exist  $\alpha_0, \beta_0, \epsilon_0$ , such that for  $\alpha \geq \alpha_0, \beta \leq \beta_0$ , and  $\epsilon \leq \epsilon_0$ , we have

$$\max_{(t, x, y) \in [\delta, T) \times \bar{B} \times \bar{B}} \Phi(t, x, y) \geq c/2 > 0. \quad (29)$$

Let  $(\bar{t}, \bar{x}, \bar{y}) \in (\delta, T) \times B \times B$  be the point where  $\Phi$  achieves the maximum. Since  $\Phi(\delta, 0, 0) \leq \Phi(\bar{t}, \bar{x}, \bar{y})$ , we get

$$\alpha|\bar{x} - \bar{y}|^2 \leq h(|\bar{x} - \bar{y}|), \quad (30)$$

in which  $h$  is the modulus of continuity of  $U$ . Since the domain is bounded,  $\alpha|\bar{x} - \bar{y}|^2 \leq K$  for some fixed constant  $K$  independent of  $\alpha, \epsilon, \beta$ . We have  $|\bar{x} - \bar{y}| \leq \sqrt{K/\alpha}$ , which implies

$$\alpha|\bar{x} - \bar{y}|^2 \leq \omega(\sqrt{\frac{K}{\alpha}}). \quad (31)$$

Denote  $\omega$  as the modulus of continuity of  $\bar{\Psi}$ . We have two cases:

1.  $U(\bar{y}, \bar{t}) = \bar{\Psi}(\bar{y}, \bar{t})$ . We have

$$\begin{aligned} W^\epsilon(\bar{x}, \bar{t}) &\leq \bar{\Psi}(\bar{x}, \bar{t}) + \frac{\epsilon}{\bar{t} - \delta} \\ &\leq \omega(|\bar{x} - \bar{y}|) + \bar{\Psi}(\bar{y}, \bar{t}) + \frac{\epsilon}{\bar{t} - \delta} \\ &= \omega(|\bar{x} - \bar{y}|) + U(\bar{y}, \bar{t}) + \frac{\epsilon}{\bar{t} - \delta}. \end{aligned}$$

Thus

$$W^\epsilon(\bar{x}, \bar{t}) - U(\bar{y}, \bar{t}) \leq \omega(|\bar{x} - \bar{y}|) + \frac{\epsilon}{\bar{t} - \delta}.$$

2.  $U(\bar{y}, \bar{t}) < \bar{\Psi}(\bar{y}, \bar{t})$ . By the same analysis as Theorem V.8.1 in [15],

$$\beta \leq \omega(\alpha|\bar{x} - \bar{y}|^2 + |\bar{x} - \bar{y}|). \quad (32)$$

Fix  $\epsilon \leq \epsilon_0, \beta \geq \beta_0$ . For each  $\alpha \leq \alpha_0$ , one of the two cases is true. If case 2 occurs infinitely many times as  $\alpha \rightarrow \infty$ , we have a contradiction, thus case 1 must occur infinitely many times as  $\alpha \rightarrow \infty$ . We have the inequality

$$\begin{aligned} &W^\epsilon(x, t) - U(x, t) - \beta(t - \delta) \\ &= \Phi(x, x, t) \leq \Phi(\bar{x}, \bar{y}, \bar{t}) \\ &\leq W^\epsilon(\bar{x}, \bar{t}) - U(\bar{y}, \bar{t}) \\ &\leq \omega(\sqrt{\frac{1}{\alpha}h(\sqrt{\frac{K}{\alpha}})}) + \frac{\epsilon}{\bar{t} - \delta}. \end{aligned}$$

Let  $\alpha \rightarrow \infty$ , then  $W^\epsilon(x, t) - U(x, t) - \beta(t - \delta) \leq \frac{\epsilon}{t - \delta}$ . Let  $\beta \rightarrow 0$ , then  $\epsilon \rightarrow 0$ , we get  $W(x, t) \leq U(x, t)$ .  $\square$

Now, combining Theorem 3 and Proposition 2, together with Lemma 10 for the  $C(\bar{Q}_T) \cap C_{loc}^{2+\alpha, 1+\frac{\alpha}{2}}(Q_T)$  solution to (13), we have

### Step III: Regularity of the (HJB) equation without jump terms

**Proposition 3.** *Assume*

1.  $a_{ij}, b_i, \Psi \in C^{0+\alpha, 0+\frac{\alpha}{2}}(\overline{Q_T})$ ,
2.  $(a_{ij})$  uniformly elliptic,
3.  $\Psi$  is semiconcave,
4.  $\Psi_t$  is bounded below, in the distributional sense,
5.  $\phi \in C^{0+\alpha, 0+\alpha/2}(\overline{Q_T})$ .

Then there exists a unique viscosity solution  $u \in W_{loc}^{(2,1),p}(Q_T) \cap C(\overline{Q_T})$  to the PDE

$$\begin{cases} \max\{u_t + Lu - f, u - \Psi\} = 0 & \text{in } Q_T, \\ u = \phi & \text{on } \partial_P Q_T, \end{cases} \quad (33)$$

for any  $p > 1$ .

Finally, since  $Mu(x, t)$  is semi-concave from Lemma 9, replacing  $\Psi(x, t)$  by  $Mu(x, t)$  gives us the regularity property of the value function.

**Theorem 4.** *Assuming  $\rho = 0$ , the value function  $V(x, t)$  is a  $W_{loc}^{2,p}(\mathbb{R}^n \times (0, T))$  viscosity solution to the (HJB) equation with  $2 \leq p < \infty$ . In particular, for each  $t \in [0, T]$ ,  $V(\cdot, t) \in C_{loc}^{1,\gamma}(\mathbb{R}^n)$  for any  $0 < \gamma < 1$ .*

In fact, if one adds additional assumption of  $a_{ij}$  and  $b_i$  in  $W_{loc}^{(2,1),\infty}(\mathbb{R}^n \times [0, T])$ , then with more detailed and somewhat tedious analysis, one can establish  $W_{loc}^{(2,1),\infty}$  regularity for the value function. For more details, see Chen [9].

## 5.2 Regularity for Value Function of First-Order Jump Diffusion

Careful checking of our analysis for the case without jumps in Section 5.1 reveals that we could relax our assumption on  $f$  to be bounded instead of Hölder continuous, and take  $f^\epsilon$  to be Hölder continuous and converge to  $f$  in  $L^\infty$ . This observation is key for the regularity with jumps.

To proceed, we will add two new assumptions in this subsection.

**Assumption 10.** *The operator  $I[\phi]$  is of order- $\delta$ , i.e., for  $(x, t)$  in any compact subset of  $\mathbb{R}^n \times [0, T]$ , there exists  $C$  such that*

$$\int_{|z|<1} |z|^\delta M(x, t, dz) < C < \infty.$$

*Remark.* Since  $\delta \in (0, 1]$ , this implies that  $\int_{|z|<1} |z| M(x, t, dz) < C$ .

**Assumption 11.** *The measure  $M(x, t, dz)$  is continuous with respect to the weighted total variation, i.e., for  $(x_n, t_n) \rightarrow (x_0, t_0)$ ,*

$$\int (|z|^\gamma + |z|^\delta) |M(x_n, t_n, dz) - M(x_0, t_0, dz)| \rightarrow 0.$$

**Proposition 4.** *With additional assumptions 10 and 11, there exists a unique  $u \in W^{(2,1),p}(Q_T)$  viscosity solution of the following equations,*

$$\begin{cases} \max\{-u_t + Lu - f - Iu, u - MV\} = 0 & \text{in } Q_T, \\ u = V(x, T - t) & \text{on } \partial_P Q_T, \end{cases} \quad (34)$$

in the following sense. For any  $\phi \in C^2(\mathbb{R}^n \times [0, T])$ ,

1. If  $u - \phi$  achieves a local maximum at  $(x_0, t_0) \in Q_T$ , then

$$\max\{-\phi_t + L\phi - f - I^0[V], u - MV\}(x_0, t_0) \leq 0;$$

2. If  $u - \phi$  achieves a local minimum at  $(x_0, t_0) \in Q_T$ , then

$$\max\{-\phi_t + L\phi - f - I^0[V], u - MV\}(x_0, t_0) \geq 0.$$

Here

$$I^0[V](x, t) = \int V(x + z, t) - V(x, t) - D\phi(x, t) \cdot z 1_{|z| < 1} \rho(dz),$$

with the boundary condition  $u = g$  on  $\mathbb{R}^n \times \{t = T\}$ .

*Proof.* Let  $\bar{b}_i = b_i - \int z_i M(x, t, dz)$  and  $\bar{f} = \int u(x + z, t) - u(x, t) M(x, t, dz)$ . It suffices to show that  $\bar{b}_i$  and  $\bar{f}$  are bounded. In fact we will show that they are continuous.

Step 1,  $\bar{f}$  is continuous:

Let  $x_n \rightarrow x_0$ , then

$$\begin{aligned} & \left| \int V(x_n + z, t) - V(x_n, t) M(x_n, t, dz) - \int V(x_0 + z, t) - V(x_0, t) M(x_0, t, dz) \right| \\ & \leq \left| \int (V(x_n + z, t) - V(x_n, t)) (M(x_n, t, dz) - M(x_0, t, dz)) \right| \\ & \quad + \left| \int (V(x_n + z, t) - V(x_n, t)) - (V(x_0 + z, t) - V(x_0, t)) M(x_0, t, dz) \right|. \end{aligned}$$

For the first term,

$$\begin{aligned} & \left| \int (V(x_n + z, t) - V(x_n, t)) (M(x_n, t, dz) - M(x_0, t, dz)) \right| \\ & \leq C \int (1 + |x_n|^\gamma + |z|^\gamma) |z|^\delta |M(x_n, t, dz) - M(x_0, t, dz)| \\ & \leq C \int |z|^\gamma + |z|^\delta |M(x_n, t, dz) - M(x_0, t, dz)| \rightarrow 0. \end{aligned}$$

For the second term, the integrand  $\rightarrow 0$  as  $n \rightarrow \infty$ . So by the dominated convergence theorem,

$$\begin{aligned} & \left| \int (V(x_n + z, t) - V(x_n, t)) - (V(x_0 + z, t) - V(x_0, t)) M(x_0, t, dz) \right| \\ & \leq \int C(1 + |x_n|^\gamma + |z|^\gamma) |z|^\delta + C(1 + |x_0|^\gamma + |z|^\gamma) |z|^\delta M(x_0, t, dz) \\ & \leq C \int |z|^\gamma + |z|^\delta M(x_0, t, dz) < \infty. \end{aligned}$$

Therefore  $\bar{f}$  is continuous in  $x$ . Now let  $t_n \rightarrow t_0$ ,

$$\begin{aligned} & \left| \int V(x + z, t_n) - V(x, t_n) M(x, t_n, dz) - \int V(x + z, t_0) - V(x, t_0) M(x, t_0, dz) \right| \\ & \leq \left| \int (V(x + z, t_n) - V(x, t_n)) (M(x, t_n, dz) - M(x, t_0, dz)) \right| \\ & \quad + \left| \int (V(x + z, t_n) - V(x, t_n)) - (V(x + z, t_0) - V(x, t_0)) M(x_0, t, dz) \right|. \end{aligned}$$

For the first term,

$$\begin{aligned} & \left| \int (V(x + z, t_n) - V(x, t_n)) (M(x, t_n, dz) - M(x, t_0, dz)) \right| \\ & \leq \int C(1 + |x|^\gamma + |z|^\gamma) |z|^\delta |M(x, t_n, dz) - M(x, t_0, dz)| \\ & \leq C \int |z|^\gamma + |z|^\delta |M(x, t_n, dz) - M(x, t_0, dz)| \rightarrow 0, \end{aligned}$$

as  $t_n \rightarrow t_0$ . For the second term, the dominated convergence theorem implies

$$\begin{aligned} & \left| \int (V(x+z, t_n) - V(x, t_n)) - (V(x+z, t_0) - V(x, t_0)) M(x_0, t, dz) \right| \\ & \leq \int C(1 + |x|^\gamma + |z|^\gamma) |z|^\delta M(x, t_0, dz) < \infty. \end{aligned}$$

Therefore  $\bar{f}$  is continuous in  $t$ .

Step 2,  $\bar{b}_i$  is continuous:

This follows easily from Assumption 11. Let  $(x_n, t_n) \rightarrow (x_0, t_0)$ ,

$$\begin{aligned} \left| \int_{|z|<1} z[M(x_n, t_n, dz) - M(x_0, t_0, dz)] \right| & \leq \int_{|z|<1} |z| |M(x_n, t_n, dz) - M(x_0, t_0, dz)| \\ & \leq \int |z|^\delta |M(x_n, t_n, dz) - M(x_0, t_0, dz)|, \end{aligned}$$

which goes to 0 as  $n \rightarrow \infty$ .

Step 3, replace  $b_i$  by  $\bar{b}_i = b_i - \int z_i M(x, t, dz)$  and  $f$  by  $\bar{f} = f + \int u(x+z, t) - u(x, t) M(x, t, dz)$ , and follow the same line of reasoning in the proof for Proposition 3.  $\square$

Notice, however, the “apparent” difference between the two types of viscosity solutions: the one in the above proposition, and the one in Theorem 2. Therefore, we need to show that the viscosity solution in Theorem 2 is also a viscosity solution of Eqn. (34). Then, with the standard local uniqueness of HJB of Eqn. (34), the regularity of value function is obtained.

**Theorem 5.**  *$V$  is also a solution of Eqn. (34), with additional assumptions 10 and 11.*

*Proof.* Suppose  $V - \phi$  has a local minimum in  $B(x_0, \theta_0) \times [t_0, t + \theta_0)$ . Then we know that

$$\max\{-\phi_t + L\phi - f - I_\theta^1[\phi] - I_\theta^2[V], V - MV\} \leq 0$$

for any  $0 < \theta < \theta_0$ . And with the additional assumptions,  $I_\theta^1[\phi] + I_\theta^2[V] \rightarrow I^0[V]$  as  $\theta \rightarrow 0$ . Therefore we have

$$\max\{-\phi_t + L\phi - f - I^0[V], V - MV\} \leq 0$$

The other inequalities can be derived similarly.  $\square$

In summary,

**Theorem 6. (Regularity of the Value Function and Uniqueness)** *With additional assumptions 10 and 11, the value function  $V(x, t)$  is a unique  $W_{loc}^{(2,1),p}(\mathbb{R}^n \times (0, T))$  viscosity solution to the (HJB) equation with  $2 \leq p < \infty$ . In particular, for each  $t \in [0, T)$ ,  $V(\cdot, t) \in C_{loc}^{1,\gamma}(\mathbb{R}^n)$  for any  $0 < \gamma < 1$ .*

## 6 Appendix

### 6.1 Appendix A: Proof of Lemma 2

*Proof.* First assume that  $\int_\tau^T H^2 d[M]$  is bounded and  $M$  is a  $L^2$ -martingale. The conclusion is clearly true when  $H$  is an elementary previsible process of the form  $\sum_i^m Z_i 1_{(S_i, T_i]}$ . Now let  $H^{(n)}$  be a sequence of such elementary previsible process that converges to  $H$  uniformly on  $\Omega \times (0, T]$ . Let  $N_t^{(n)} = \int_\tau^t H_s^{(n)} dM_s$ , and  $N_t = \int_\tau^t H_s dM_s$ .

First we show that quadratic variation is preserved under regular conditional probability distribution. By Lemma 1,  $Q_t = M_t - M_{t \wedge \tau}$  is a martingale. Consider the quadratic variation  $[Q]$  under  $\mathbb{P}$ . By

definition,  $Q_t^2 - [Q]_t$  is a martingale. Thus by Lemma 1, for almost every  $\omega$ ,  $Q_t^2 - [Q]_t - (Q_{t \wedge \tau}^2 - [Q]_{t \wedge \tau})$  is martingale under  $(\mathbb{P}|\mathcal{F}_\tau)(\omega)$ .

$$\begin{aligned} & Q_t^2 - [Q]_t - (Q_{t \wedge \tau}^2 - [Q]_{t \wedge \tau}) \\ &= (M_t - M_{t \wedge \tau})^2 - ([M]_t - [M]_{t \wedge \tau}) \\ &\quad - ((M_{t \wedge \tau} - M_{t \wedge \tau})^2 - ([M]_{t \wedge \tau} - [M]_{t \wedge \tau})) \\ &= Q_t^2 - ([M]_t - [M]_{t \wedge \tau}) \end{aligned}$$

This shows that under  $(\mathbb{P}|\mathcal{F}_\tau)(\omega)$ ,  $[Q]_t^{(\mathbb{P}|\mathcal{F}_\tau)(\omega)} = [M]_t - [M]_{t \wedge \tau}$ . This allows us to simply write  $[\cdot]$  instead of  $[\cdot]^{(\mathbb{P}|\mathcal{F}_\tau)(\omega)}$ . Hence,

$$\int_\tau^T H^2 d[M] = \int_\tau^T H^2 d[N]$$

is bounded under  $(\mathbb{P}|\mathcal{F}_\tau)(\omega)$ .

By the definition of  $H^{(n)}$ ,

$$\begin{aligned} & \mathbb{E}^\mathbb{P} \left[ \liminf_n \mathbb{E}^{(\mathbb{P}|\mathcal{F}_\tau)(\omega)} \left[ \int_\tau^T |H_s^{(n)} - H_s|^2 d[Q]_s \right] \right] \\ & \leq \liminf_n \mathbb{E}^\mathbb{P} \left[ \mathbb{E}^{(\mathbb{P}|\mathcal{F}_\tau)(\omega)} \left[ \int_\tau^T |H_s^{(n)} - H_s|^2 d[Q]_s \right] \right] \\ & = \liminf_n \mathbb{E}^\mathbb{P} \left[ \int_\tau^T |H_s^{(n)} - H_s|^2 d[M]_s \right]^2 = 0. \end{aligned}$$

Thus,  $\liminf_n \mathbb{E}^{(\mathbb{P}|\mathcal{F}_\tau)(\omega)} \left[ \int_\tau^T |H_s^{(n)} - H_s|^2 d[Q]_s \right] = 0$  for almost every  $\omega$ . On the other hand,

$$\begin{aligned} & \mathbb{E}^\mathbb{P} \left[ \liminf_n \mathbb{E}^{(\mathbb{P}|\mathcal{G})(\omega)} [N_T^n - N_T]^2 \right] \\ & \leq \liminf_n \mathbb{E}^\mathbb{P} \left[ \mathbb{E}^{(\mathbb{P}|\mathcal{G})(\omega)} [N_T^n - N_T]^2 \right] \\ & \leq \liminf_n \mathbb{E}^\mathbb{P} [N_T^n - N_T]^2 = 0. \end{aligned}$$

So we have

$$\liminf_n \mathbb{E}^{(\mathbb{P}|\mathcal{G})(\omega)} [(N^n - N)_T]^2 = 0,$$

for  $\mathbb{P}$ -a.e.  $\omega$ . This proves the claim. The general case follows from the localization technique.  $\square$

## 6.2 Appendix B: Proof of Lemma 11

*Proof.* Define the operator  $A : C^{2+\alpha, 1+\alpha/2}(\overline{Q}_T) \rightarrow C^{2+\alpha, 1+\alpha/2}(\overline{Q}_T)$  by the following:  $A[v] = u$  is the solution to the PDE

$$\begin{cases} u_t + Lu + \beta_\epsilon(v - \overline{\Psi}) = 0 & \text{on } Q_T, \\ u = 0 & \text{on } \partial_P Q_T. \end{cases} \quad (35)$$

By the Schauder's estimates (Theorem 4.28 in [24]), we have,

$$\begin{aligned} \|u\|_{C^{2+\alpha, 1+\alpha/2}(\overline{Q}_T)} & \leq C \|\beta_\epsilon(v - \overline{\Psi})\|_{C^{\alpha, \alpha/2}(\overline{Q}_T)} \\ & \leq C_\epsilon \|v - \overline{\Psi}\|_{C^{\alpha, \alpha/2}(\overline{Q}_T)}. \end{aligned}$$

Thus the map  $A$  is clearly continuous and compact.

The next step is to show that the set  $\{u : u = \lambda A[u], 0 \leq \lambda \leq 1\}$  is bounded. Then we can apply Schaefer's Fixed Point Theorem (Theorem 9.4 in ([13])). Suppose  $u = \lambda A[u]$  for some  $0 \leq \lambda \leq 1$ . Then,

$$\begin{cases} u_t + Lu + \lambda \beta_\epsilon(u - \bar{\Psi}) = 0 & \text{on } Q_T, \\ u = 0 & \text{on } \partial_P Q_T. \end{cases} \quad (36)$$

Since

$$\begin{aligned} \|u\|_{C^{2+\alpha, 1+\alpha/2}(\bar{Q}_T)} &\leq C_\epsilon \|v - \bar{\Psi}\|_{C^{\alpha, \alpha/2}(\bar{Q}_T)} \\ &\leq C_\epsilon (\|u\|_{C^{\alpha, \alpha/2}(\bar{Q}_T)} + \|\bar{\Psi}\|_{C^{\alpha, \alpha/2}(\bar{Q}_T)}) \\ &\leq C_\epsilon (\|u\|_{C(\bar{Q}_T)}^{1/2} \|u\|_{C^{2+\alpha, 1+\alpha/2}(\bar{Q}_T)}^{1/2} + \|\bar{\Psi}\|_{C^{\alpha, \alpha/2}(\bar{Q}_T)}) \\ &\leq \frac{1}{2} \|u\|_{C^{2+\alpha, 1+\alpha/2}(\bar{Q}_T)} + C_\epsilon (\|u\|_{C(\bar{Q}_T)} + \|\bar{\Psi}\|_{C^{\alpha, \alpha/2}(\bar{Q}_T)}). \end{aligned}$$

Thus

$$\|u\|_{C^{2+\alpha, 1+\alpha/2}(\bar{Q}_T)} \leq C_\epsilon (\|u\|_{C(\bar{Q}_T)} + \|\bar{\Psi}\|_{C^{\alpha, \alpha/2}(\bar{Q}_T)}). \quad (37)$$

So we only need to bound  $u$  independent of  $\lambda$  now.

If  $\lambda = 0$ , then  $u = 0$ . So we can assume that  $\lambda > 0$ . Suppose  $u$  has a maximum at  $(x_0, t_0) \in Q_T$ . Then,  $-\lambda \beta_\epsilon(u(x_0, t_0) - \bar{\Psi}(x_0, t_0)) = (u_t + Lu)(x_0, t_0) \geq 0$ ,  $\beta_\epsilon(u(x_0, t_0) - \bar{\Psi}(x_0, t_0)) \leq 0$ ,  $u(x_0, t_0) \leq \bar{\Psi}(x_0, t_0)$ , and  $u = 0$  on  $\partial_P Q_T$ . So we get  $u \leq \|\bar{\Psi}\|_{L^\infty(Q_T)}$ .

For a lower bound, consider the open set  $\Omega = \{u < \bar{\Psi}\}$  in  $\bar{Q}_T$ . Since in  $\Omega$ ,  $u_t + Lu \geq 0$ ,  $u \geq \inf_{\partial_P \Omega} u$ . Yet,  $\partial_P \Omega \subset \partial_P Q_T \cup \{u \geq \bar{\Psi}\}$ , and in both cases  $u$  is bounded below. Thus we conclude that  $u$  is bounded independently of  $\lambda$ , and  $\|u\|_{L^\infty(Q_T)} \leq \|\bar{\Psi}\|_{L^\infty(Q_T)}$ .

Now Schaefer's Fixed Point Theorem (Theorem 9.4 in [13]) gives us the existence of  $u \in C^{2+\alpha, 1+\alpha/2}(\bar{Q}_T)$  that solves (15). Now we have  $-\beta_\epsilon(u - \bar{\Psi}) \in C^{2+\alpha, 1+\alpha/2}(\bar{Q}_T)$ . By the Schauder's estimates again, we have  $u \in C^{4+\alpha, 2+\alpha/2}(\bar{Q}_T)$ .  $\square$

## References

- [1] G. BARLES, E. CHASSEIGNE, AND C. IMBERT, *On the dirichlet problem for second-order elliptic integro-differential equations*, Indiana University Mathematics Journal, 57 (2008), pp. 213–246.
- [2] E. BAYRAKTAR AND H. XING, *Analysis of the optimal exercise boundary of american options for jump diffusions*, SIAM Journal on Mathematical Analysis, 41(2) (2009), pp. 825–860.
- [3] A. BENSOUSSAN AND J.-L. LIONS, *Impulse Control and Quasivariational Inequalities*, Bordas, (1982). Translation of *Contrôle Impulsionnel et Inéquations Quasi-variationnelles*.
- [4] T. BIELECKI AND S. PLISKA, *Risk sensitive asset management with fixed transaction costs*, Finance and Stochastics, 4 (2000), pp. 1–33.
- [5] B. BOUCHARD AND M. NUTZ, *Weak dynamic programming for generalized state constraints*, Preprint, (2011).
- [6] B. BOUCHARD AND N. TOUZI, *Weak dynamic programming principle for viscosity solutions*, SIAM J. Control Optim., 49(3) (2011), pp. 948–962.
- [7] A. CADENILLAS, T. CHOULLI, M. TAKSAR, AND L. ZHANG, *Classical and impulse stochastic control for the optimization of the dividend and risk policies of an insurance firm*, Mathematical Finance, 16 (2006), pp. 181–202.
- [8] A. CADENILLAS AND F. ZAPATERO, *Optimal central bank intervention in the foreign exchange market*, Journal of Econ. Theory, 97 (1999), pp. 218–242.
- [9] Y. A. CHEN, *Some Control Problems on Multi-Dimensional Jump Diffusions*, Ph.D. Dissertation, Dept. of Math, U.C. Berkeley, (2012).

- [10] M. A. DAVIS, X. GUO, AND G. L. WU, *Impulse controls for multi-dimensional jump diffusions*, SIAM J. Control Optim., 48 (2010), pp. 5276–5293.
- [11] J. E. EASTHAM AND K. J. HASTINGS, *Optimal impulse control of portfolios*, Mathematics of Operations Research, 13 (1988), pp. 588–605.
- [12] N. ELKOURI, *Les aspects probabilistes du controle stochastique*, Lecture Notes in Math. Springer, New York, 876 (1981).
- [13] L. C. EVANS, *Partial Differential Equations*, AMS, (1998).
- [14] I. EVSTIGNEEV, *Measurable selection and dynamic programming*, Mathematics of Operations Research, 1 No. 3 (1976), pp. 267–272.
- [15] W. H. FLEMING AND H. M. SONER, *Controlled Markov Processes and Viscosity Solutions*, Springer, New York, Second Edition, (2006).
- [16] A. FRIEDMAN, *Variational Principles and Free-Boundary Problems*, John Wiley & Sons, (1982).
- [17] X. GUO AND G. L. WU, *Smooth fit principle for impulse control of multidimensional diffusion processes*, SIAM J. Control Optim., 48 (2009), pp. 594–617.
- [18] Y. ISHIKAWA, *Optimal control problem associated with jump processes*, Applied Mathematics and Optimization, 50 (2004), pp. 21–65.
- [19] M. JEANBLANC AND S. SHIRYAYEV, *Optimization of the flow of dividends*, Russian Math. Surveys, 50 (1995), pp. 257–277.
- [20] M. JEANBLANC-PICQUÉ, *Impulse control method and exchange rate*, Mathematical Finance, 3 (1993), pp. 161–177.
- [21] O. KALLENBERG, *Foundations of Modern Probability*, Springer-Verlag New York, Second Edition, (2002).
- [22] R. KORN, *Portfolio optimization with strictly positive transaction costs and impulse control*, Finance and Stochastics, 2 (1998), pp. 85–114.
- [23] R. KORN, *Some applications of impulse control in mathematical finance*, Math. Meth. Oper. Res., 50 (1999), pp. 493–518.
- [24] G. M. LIEBERMAN, *Second Order Parabolic Differential Equations*, World Scientific Publishing Co. Inc., River Edge, NJ, (1996).
- [25] V. LYVATH, M. MNIF, AND H. PHAM, *A model of optimal portfolio selection under liquidity risk and price impact*, Finance and Stochastics, (2007), pp. 51–90.
- [26] D. C. MAUER AND A. TRIANTIS, *Interactions of corporate financing and investment decisions: a dynamic framework*, Journal of Finance, 49 (1994), pp. 1253–1277.
- [27] A. J. MORTON AND S. PLISKA, *Optimal portfolio management with fixed transaction costs*, Mathematical Finance, 5 (1995), pp. 337–356.
- [28] G. MUNDACA AND B. ØKSENDAL, *Optimal stochastic intervention control with application to the exchange rate*, J. of Mathematical Economics, 29 (1998), pp. 225–243.
- [29] B. ØKSENDAL AND A. SULEM, *Optimal consumption and portfolio with both fixed and proportional transaction costs*, SIAM J. Cont. Optim., 40 (2002), pp. 1765–1790.
- [30] ———, *Applied Stochastic Control of Jump Diffusions*, Universitext. Springer-Verlag, Berlin, (2004).
- [31] H. PHAM, *Optimal stopping of controlled jump diffusion processes: a viscosity solution approach*, Journal of Mathematical Systems, Estimation, and Control, 8(1) (1998), pp. 1–27.



- [32] R. C. SEYDEL, *Existence and uniqueness of viscosity solutions for QVI associated with impulse control of jump diffusions*, Stochastic Processes and their Applications, (2009), pp. 3719–3748.
- [33] D. W. STROOCK AND S. VARADHAN, *Multidimensional Diffusion Processes*, Springer-Verlag Heidelberg, (2006).
- [34] S. J. TANG AND J. M. YONG, *Finite horizon stochastic optimal switching and impulse controls with a viscosity solution approach*, Stochastics Stochastics Rep., 45(3-4) (1993), pp. 145–176.
- [35] A. TRIANTIS AND J. E. HODDER, *Valuing flexibility as a complex option*, Journal of Finance, 45 (1990), pp. 549–565.
- [36] J. M. YONG AND X. Y. ZHOU, *Stochastic Controls: Hamiltonian Systems and HJB Equations*, Springer-Verlag New York, (1999).